Commun. Math. Phys. 255, 61–95 (2005)
Digital Object Identifier (DOI) 10.1007/s00220-004-1252-y

Communications in
Mathematical Physics

Tau Functions for the Dirac Operator on the Cylinder

O. Lisovyy[1,2]

[1] Bogolyubov Institute for Theoretical Physics, Metrolohichna str., 14-b, Kyiv-143, 03143 Kiev, Ukraine
[2] Département de Mathématiques, Université d'Angers, 2 Boulevard Lavoisier, 49045 Angers, France

Received: 6 February 2004 / Accepted: 6 July 2004
Published online: 11 January 2005 –

Abstract: The goal of the present paper is to calculate the determinant of the Dirac operator with a mass in the cylindrical geometry. The domain of this operator consists of functions that realize a unitary one-dimensional representation of the fundamental group of the cylinder with n marked points. The determinant represents a version of the isomonodromic τ-function, introduced by M. Sato, T. Miwa and M. Jimbo. It is calculated by comparison of two sections of the det*-bundle over an infinite-dimensional grassmannian. The latter is composed of the spaces of boundary values of some local solutions to the Dirac equation. The principal ingredients of the computation are the formulae for the Green function of the singular Dirac operator and for the so-called canonical basis of global solutions on the 1-punctured cylinder. We also derive a set of deformation equations satisfied by the expansion coefficients of the canonical basis in the general case and find a more explicit expression for the τ-function in the simplest case $n = 2$.

1. Introduction

The main objective of quantum field theory is the calculation of correlation functions of local operators, usually represented via functional integrals

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \frac{\int \mathcal{D}\varphi\ \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)e^{S[\varphi]}}{\int \mathcal{D}\varphi\ e^{S[\varphi]}}\ .$$

For a generic interacting QFT such calculation can be done only by means of perturbation theory. However, in two dimensions there is an interesting way to construct an interacting theory from the free one. Let us consider the action of free massive Dirac fermions in the flat spacetime,

$$S[\psi,\bar{\psi}] = \int d^2x\,\bar{\psi}D\psi.$$

Correlation functions of the (interacting) monodromy fields are defined as

$$\langle \mathcal{O}^{\lambda_1}(a_1)\dots\mathcal{O}^{\lambda_n}(a_n)\rangle = \frac{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{\int d^2x\; \bar{\psi}D^{a,\lambda}\psi}}{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{\int d^2x\; \bar{\psi}D\psi}}\,. \tag{1.1}$$

The integration in the numerator is performed over field configurations that are branched at the points a_ν with the monodromies $e^{2\pi i\lambda_\nu}$ ($\nu = 1,\dots,n$). This change of domain of the Dirac operator is symbolically reflected by indexing $D^{a,\lambda}$. The integrals in (1.1) can be formally evaluated to the determinants of the corresponding operators,

$$\langle \mathcal{O}^{\lambda_1}(a_1)\dots\mathcal{O}^{\lambda_n}(a_n)\rangle = \frac{\det D^{a,\lambda}}{\det D}\,. \tag{1.2}$$

Note, however, that the RHSs of both (1.1) and (1.2) are equally ill-defined quantities.

The determinants of Dirac operators on compact manifolds are usually determined via the ζ-function regularization. Starting from [16], they have been extensively studied in the mathematical literature. The massless Dirac operators on Riemann surfaces deserve special attention, as multiloop contributions to the partition function in the string theory are expressed through their determinants (rigorously defined by D. Quillen in [15]). These determinants can be thought of as the functions on the moduli space of complex structures on the surface.

In the case we are interested in the Dirac operator is defined not on a compact manifold, but on the universal covering of a surface with marked points, the determinant being the function of their positions. The problem of rigorous definition of the determinant and the Green function for the Dirac operator with branching points on the Euclidean plane was solved by Palmer in [10]. His work relies heavily on the analysis of monodromy preserving deformations for the Dirac operator, developed earlier by Sato, Miwa and Jimbo [17]. More precisely, Palmer's determinant represents another version of the SMJ τ-function. Its logarithmic derivatives with respect to the coordinates of branching points are expressed via the expansion coefficients of some special solutions to the Dirac equation, that can be constructed from the so-called canonical basis of solutions. The theory of isomonodromic deformations gives a set of nonlinear differential equations satisfied by these expansion coefficients. Moreover, in the simplest case $n = 2$ an explicit formula for the determinant was found [10].

Later similar results were obtained for the Dirac operator on the Poincaré disk [8, 12, 13]. In this connection we should also mention the recent work of Doyon [5], where the two-point correlation function of monodromy fields in the hyperbolic geometry was calculated by field-theoretic methods.

In the present paper, we define and calculate the determinant of the massive Dirac operator in cylindrical geometry. The latter corresponds to QFT in the finite volume or at non-zero temperature. This work was inspired by recent progress in the study of the Ising model — calculation of finite-size correlation functions [1, 2], spin matrix elements [3] and direct derivation of the differential equations satisfied by the two-point correlator in the continuum limit [7]. The Ising model is related to the above theory with special monodromy $\lambda_\nu = \pm\frac{1}{2}$.

This paper is organized as follows. In the next section we introduce the canonical basis of global solutions to the Dirac equation on the cylinder and calculate it explicitly for $n = 1$ (see Theorem 2.3). All subsequent computations are based on these formulae. The Green function of the singular Dirac operator is defined in Sect. 3. Its derivatives with respect to the coordinates of branching points are expressed through some solutions to the Dirac equation and have remarkable factorized form (formulae (3.23) and (3.24)). At the end of the section, the Green function on the 1-punctured cylinder is computed (see (3.29)–(3.30)). Section 4 is devoted to the definition and calculation of the τ-function. We introduce the det*-bundle over an infinite-dimensional grassmannian that consists of the spaces of boundary values of some local solutions to the Dirac equation. The τ-function is obtained by comparison of the canonical section of this det*-bundle with a section that is constructed using the one-point Green functions. The logarithmic derivatives of the τ-function can also be written in terms of the expansion coefficients of the solutions (3.20). This shows that it is independent of the chosen localization. As an illustration, we find a more explicit expression for the τ-function when $n = 2$ (formula (4.25)). Finally, in Sect. 5 a set of deformation equations for the expansion coefficients is derived. We conclude with a brief discussion of possible generalizations, open problems and application of obtained results in quantum field theory at non-zero temperature.

2. Canonical Basis of Solutions to Dirac Equation

2.1. Definitions. Let $a = (a_1, \dots, a_n)$ be a collection of n distinct points on the cylinder $\mathcal{C}$. The fundamental group $\pi_1(\mathcal{C}\backslash a; x_0)$ is generated by homotopy classes of $n+1$ loops $\gamma_0, \dots, \gamma_n$ shown in Fig. 1. It acts on the universal covering $\widetilde{\mathcal{C}\backslash a}$ by deck transformations. Let us fix a one-dimensional unitary representation

$$\begin{aligned} &\rho_\lambda : \pi_1(\mathcal{C}\backslash a; x_0) \to U(1), \qquad [\gamma_\nu] \mapsto e^{-2\pi i \lambda_\nu}, \qquad \nu = 0, \dots, n, \\ &\lambda_0 \in \mathbb{R}, \qquad \lambda_\nu \in \mathbb{R}\backslash\mathbb{Z}, \qquad \nu = 1, \dots, n. \end{aligned} \tag{2.1}$$

As usual, we replace the cylinder by the strip $S = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le \beta\}$ whose upper and lower edges are identified. The Dirac operator on $\widetilde{\mathcal{C}\backslash a}$ is induced by the Dirac operator on $\mathbb{R}^2$, which can be written as

$$D = \begin{pmatrix} \frac{m}{2} & -\partial_z \\ -\partial_{\bar z} & \frac{m}{2} \end{pmatrix}, \tag{2.2}$$

where $z, \bar z$ — standard complex coordinates

$$\begin{cases} z = x + iy, \\ \bar z = x - iy, \end{cases} \qquad \begin{cases} \partial_z = \frac{1}{2}(\partial_x - i\partial_y), \\ \partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y). \end{cases}$$

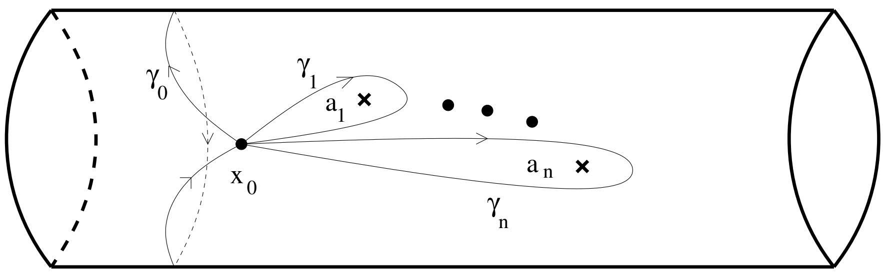

Fig. 1

We are looking for multivalued solutions $\tilde{\psi} : \widetilde{\mathcal{C}\backslash a} \to \mathbb{C}^2$ to Dirac equation that transform according to the representation (2.1),

$$D\tilde{\psi}(x) = 0, \qquad \tilde{\psi}(\gamma x) = \rho_\lambda([\gamma]) \cdot \tilde{\psi}(x).$$

This problem can be reformulated as follows. Fix a system of branchcuts $b = (b_1, \ldots, b_n; d_0, \ldots, d_n)$ shown in Fig. 2 and consider the solutions to Dirac equation on $\mathcal{C}\backslash b$ that can be continued across the branchcuts away from the points $a_1, \ldots, a_n$. The solutions we are interested in have left and right continuations across b_ν that differ by the factor $e^{2\pi i\lambda_\nu}$ ($\nu = 1, \ldots, n$). The continuations across d_ν differ by $\exp\big(2\pi i \sum_{k=0}^{\nu} \lambda_k\big)$, $\nu = 0, \ldots, n$.

To describe the local behaviour of such solutions in the neighbourhood of the point a_ν, consider an open disk B of sufficiently small but finite radius, centered at a_ν, and introduce in B polar coordinates

$$\begin{cases} r = |z - a_\nu|^{1/2}, \\ \varphi = \frac{1}{2i} \ln \frac{z-a_\nu}{\bar{z}-\bar{a}_\nu}, \end{cases} \qquad \begin{cases} \partial_z = \frac{1}{2}\, e^{-i\varphi}(\partial_r - \frac{i}{r}\,\partial_\varphi), \\ \partial_{\bar{z}} = \frac{1}{2}\, e^{i\varphi}(\partial_r + \frac{i}{r}\,\partial_\varphi). \end{cases}$$

The local form of the Dirac operator on B is then

$$D = \frac{1}{2}\begin{pmatrix} m & -e^{-i\varphi}\big(\partial_r - \frac{i}{r}\,\partial_\varphi\big) \\ -e^{i\varphi}\big(\partial_r + \frac{i}{r}\,\partial_\varphi\big) & m \end{pmatrix}.$$

Since for any multivalued solution ψ the function $e^{-i\lambda_\nu\varphi}\psi$ is single-valued on B, it can be expanded in Fourier series. Substituting the series into Dirac equation, one obtains [17]

$$\psi[a_\nu] = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \left\{a_k w_{k+\lambda_\nu}[a_\nu] + b_k w^*_{k-\lambda_\nu}[a_\nu]\right\}, \tag{2.3}$$

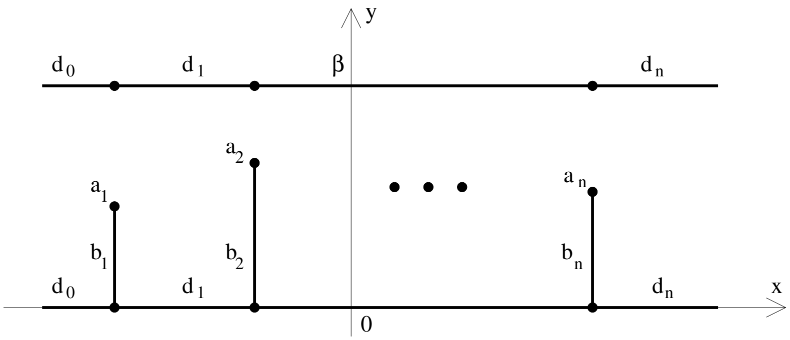

Fig. 2

where

$$w_l[a_\nu] = \begin{pmatrix} e^{i(l-1/2)\varphi} I_{l-1/2}(mr) \\ e^{i(l+1/2)\varphi} I_{l+1/2}(mr) \end{pmatrix}, \qquad w_l^*[a_\nu] = \begin{pmatrix} e^{-i(l+1/2)\varphi} I_{l+1/2}(mr) \\ e^{-i(l-1/2)\varphi} I_{l-1/2}(mr) \end{pmatrix}, \tag{2.4}$$

and $I_l(x)$ is the modified Bessel function of the first kind.

To obtain some kind of regularity, we put certain conditions on the singular behaviour of the function ψ at the point a_ν. There two essential types of constraints:

- Let $0 < \lambda_\nu < 1$ and require ψ to be square integrable in the neighborhood of a_ν. When $|z| \to a_\nu$, the asymptotics of special solutions has the form

$$w_l[a_\nu] \sim \begin{pmatrix} \frac{(m(z-a_\nu)/2)^{l-\frac{1}{2}}}{(l-\frac{1}{2})!} \\ \frac{(m(z-a_\nu)/2)^{l+\frac{1}{2}}}{(l+\frac{1}{2})!} \end{pmatrix} + \ldots, \qquad w_l^*[a_\nu] \sim \begin{pmatrix} \frac{(m(\bar{z}-\bar{a}_\nu)/2)^{l+\frac{1}{2}}}{(l+\frac{1}{2})!} \\ \frac{(m(\bar{z}-\bar{a}_\nu)/2)^{l-\frac{1}{2}}}{(l-\frac{1}{2})!} \end{pmatrix} + \ldots,$$

  where factorials are understood as $l! = \Gamma(l+1)$. Then to satisfy the condition of square integrability, a part of coefficients in (2.3) must vanish,

$$\psi[a_\nu] = a_{-1/2} w_{-1/2+\lambda_\nu}[a_\nu] + \sum_{k>0} \left\{ a_k w_{k+\lambda_\nu}[a_\nu] + b_k w^*_{k-\lambda_\nu}[a_\nu] \right\}. \tag{2.5}$$

- Let $-\frac{1}{2} < \lambda_\nu < \frac{1}{2}$ and require

$$\begin{pmatrix} (z-a_\nu)^{-\lambda_\nu} & 0 \\ 0 & (\bar{z}-\bar{a}_\nu)^{\lambda_\nu} \end{pmatrix} \psi \in H^1[a_\nu], \tag{2.6}$$

  where $H^1[a_\nu]$ denotes the space of functions that are single-valued and square integrable in the neighborhood of a_ν together with their first derivatives. Then

$$\psi[a_\nu] = \sum_{k>0} \left\{ a_k w_{k+\lambda_\nu}[a_\nu] + b_k w^*_{k-\lambda_\nu}[a_\nu] \right\}. \tag{2.7}$$

Now let us consider multivalued solutions with monodromy (2.1) that are square integrable at $|x| \to \infty$ and satisfy (2.5) or (2.7) in the neighborhood of *each* singularity. The spaces of solutions of the first and second type will be denoted by $\mathbf{W}^{a,\lambda}$ and $\widetilde{\mathbf{W}}^{a,\lambda}$ respectively.

Theorem 2.1. $\dim \mathbf{W}^{a,\lambda} \le n$; $\dim \widetilde{\mathbf{W}}^{a,\lambda} = 0$.

■ Consider a positive definite scalar product on $\mathbf{W}^{a,\lambda}$:

$$\langle u, w \rangle = \overline{\langle w, u \rangle} = \frac{m^2}{2} \int_{\mathcal{C}\backslash a} \bar{u} \cdot w \, i dz \wedge d\bar{z} = \frac{m^2}{2} \int_{\mathcal{C}\backslash a} (\bar{u}_1 w_1 + \bar{u}_2 w_2) \, i dz \wedge d\bar{z}. \tag{2.8}$$

Note that the expression under the integral is indeed a single-valued function on $\mathcal{C}\backslash a$. This function is integrable due to imposed boundary conditions. From the Dirac equation on $\mathcal{C}\backslash b$ it follows that

$$\begin{cases} \frac{m}{2} w_1 = \partial_z w_2, \\ \frac{m}{2} w_2 = \partial_{\bar{z}} w_1, \end{cases} \qquad \begin{cases} \frac{m}{2} \bar{u}_1 = \partial_{\bar{z}} \bar{u}_2, \\ \frac{m}{2} \bar{u}_2 = \partial_z \bar{u}_1, \end{cases}$$

and we get

$$\frac{m}{2}\,(\bar{u}_1 w_1 + \bar{u}_2 w_2)\,dz\wedge d\bar{z} = -d(\bar{u}_2 w_1 dz) = d(\bar{u}_1 w_2 d\bar{z}). \tag{2.9}$$

Denote by $D_\varepsilon(a_\nu)$ the disk of radius ε about a_ν. Using (2.9) and the Stokes theorem, one obtains

$$\begin{aligned}\langle u, w\rangle &= im\sum_{\nu=1}^{n}\lim_{\varepsilon\to 0}\oint\limits_{\partial D_\varepsilon(a_\nu)} \bar{u}_2 w_1 dz\\ &= im\sum_{\nu=1}^{n}\lim_{\varepsilon\to 0}\oint\limits_{\partial D_\varepsilon(a_\nu)}\left(\overline{a^{(\nu)}_{-1/2}(u)}\,\frac{(m\bar{z}/2)^{\lambda_\nu}}{\lambda_\nu!}+\ldots+\overline{b^{(\nu)}_{1/2}(u)}\,\frac{(mz/2)^{-\lambda_\nu}}{(-\lambda_\nu)!}+\ldots\right)\\ &\quad\times\left(a^{(\nu)}_{-1/2}(w)\,\frac{(mz/2)^{\lambda_\nu-1}}{(\lambda_\nu-1)}+\ldots+b^{(\nu)}_{1/2}(w)\,\frac{(m\bar{z}/2)^{1-\lambda_\nu}}{(1-\lambda_\nu)}+\ldots\right)dz\\ &= -4\sum_{\nu=1}^{n}\overline{b^{(\nu)}_{1/2}(u)}\,a^{(\nu)}_{-1/2}(w)\sin\pi\lambda_\nu. \end{aligned}\tag{2.10}$$

Or, analogously

$$\begin{aligned}\langle u, w\rangle &= -im\sum_{\nu=1}^{n}\lim_{\varepsilon\to 0}\oint\limits_{\partial D_\varepsilon(a_\nu)} \bar{u}_1 w_2 d\bar{z}\\ &= -im\sum_{\nu=1}^{n}\lim_{\varepsilon\to 0}\oint\limits_{\partial D_\varepsilon(a_\nu)}\left(\overline{a^{(\nu)}_{-1/2}(u)}\,\frac{(m\bar{z}/2)^{\lambda_\nu-1}}{(\lambda_\nu-1)!}+\ldots+\overline{b^{(\nu)}_{1/2}(u)}\,\frac{(mz/2)^{1-\lambda_\nu}}{(1-\lambda_\nu)!}+\ldots\right)\\ &\quad\times\left(a^{(\nu)}_{-1/2}(w)\,\frac{(mz/2)^{\lambda_\nu}}{\lambda_\nu!}+\ldots+b^{(\nu)}_{1/2}(w)\,\frac{(m\bar{z}/2)^{-\lambda_\nu}}{(-\lambda_\nu)!}+\ldots\right)d\bar{z}\\ &= -4\sum_{\nu=1}^{n}\overline{a^{(\nu)}_{-1/2}(u)}\,b^{(\nu)}_{1/2}(w)\sin\pi\lambda_\nu = \overline{\langle w, u\rangle}. \end{aligned}\tag{2.11}$$

If the dimension of $\mathbf{W}^{a,\lambda}$ were greater than n, we would be able to construct a solution $v\in\mathbf{W}^{a,\lambda}$ with all $a^{(\nu)}_{-1/2}(v)=0$ ($\nu=1,\ldots,n$). This solution has zero norm $\langle v, v\rangle = 0$ and, therefore, vanishes identically, implying the first statement of the theorem.

Note that the solutions of the second type are square integrable with respect to the inner product (2.8). We can show in absolutely analogous fashion that $\langle v, v\rangle = 0$ for all $v\in\widetilde{\mathbf{W}}^{a,\lambda}$. Consequently, $\dim\widetilde{\mathbf{W}}^{a,\lambda}=0$. □

Suppose[1] that $\dim\mathbf{W}^{a,\lambda}=n$. Then we can fix a *canonical basis* $\{\mathbf{w}_\mu\}_{\mu=1,\ldots,n}$ of this space, having chosen $a^{(\nu)}_{-1/2}(\mathbf{w}_\mu)=\delta_{\mu\nu}$:

$$\mathbf{w}_\mu[a_\nu] = \delta_{\mu\nu}w_{-1/2+\lambda_\nu}[a_\nu] + \sum_{k>0}\left\{a^{(\nu)}_{k}(\mathbf{w}_\mu)w_{k+\lambda_\nu}[a_\nu] + b^{(\nu)}_{k}(\mathbf{w}_\mu)w^{*}_{k-\lambda_\nu}[a_\nu]\right\}. \tag{2.12}$$

[1] The proof is based on some technique from functional analysis and is very close to the proof of Theorem 3.2.4 in [17].

Remark. Let us calculate the inner product of two elements of the canonical basis in two ways — by the formula (2.10) and its "conjugate" (2.11):

$$\langle \mathbf{w}_\mu, \mathbf{w}_\nu \rangle = -4\overline{b^{(\nu)}_{1/2}(\mathbf{w}_\mu)} \sin \pi\lambda_\nu = -4b^{(\mu)}_{1/2}(\mathbf{w}_\nu) \sin \pi\lambda_\mu. \tag{2.13}$$

We have obtained a set of algebraic relations between the expansion coefficients $b^{(\nu)}_{1/2}(\mathbf{w}_\mu)$. In what follows, we will deduce additional relations and use them in the construction of deformation equations.

The "planar" analog of the previous theorem has an instructive illustration when $n = 1$. In the case of a single branching point one can suppose it to lie at zero. Then any solution with required singular behaviour is represented by the expansion

$$\psi = a_{-1/2} w_{-1/2+\lambda}[0] + \sum_{k>0} \left\{a_k w_{k+\lambda}[0] + b_k w^*_{k-\lambda}[0]\right\}$$

on the whole punctured plane $\mathbb{R}^2 \backslash \{0\}$. This expansion will be square integrable at infinity if and only if

$$\begin{cases} a_k = 0 \text{ for } k > 0, \\ b_k = 0 \text{ for } k > 1, \\ b_{1/2} = -a_{-1/2}, \end{cases}$$

since the only integrable combinations of partial solutions (2.4) are

$$\hat{w}_l[0] = w^*_{-l}[0] - w_l[0].$$

Then, as one could expect, for $n = 1$ the space $\mathbf{W}^{0,\lambda}$ is generated by the single element of canonical basis

$$\mathbf{w} = w_{-1/2+\lambda}[0] - w_{1/2-\lambda}[0] = -\hat{w}_{1/2-\lambda}[0].$$

With some effort, it is also possible to find an explicit formula for the canonical basis on the 1-punctured cylinder. This problem will be solved in the next subsection, using some generalization of the method of Fonseca and Zamolodchikov [6].

2.2. Canonical basis on the cylinder with one branching point. We are looking for the solution ψ to Dirac equation on the strip $0 < y < \beta$,

$$\begin{pmatrix} \frac{m}{2} & -\partial_z \\ -\partial_{\bar{z}} & \frac{m}{2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0,$$

which has the following properties:

- The continuations of this solution to the left and right halfplane are quasiperiodic in y,

$$\psi(x, y + \beta) = e^{2\pi i\lambda_0} \psi(x, y) \text{ for } x < 0, \tag{2.14}$$

$$\psi(x, y + \beta) = e^{2\pi i\tilde{\lambda}}\ \psi(x, y) \text{ for } x > 0, \tag{2.15}$$

  where $\tilde{\lambda} = \lambda_0 + \lambda_1$.

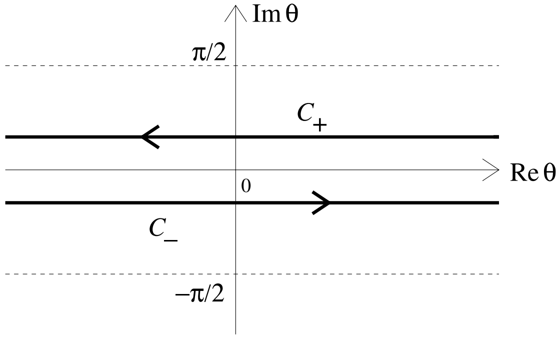

Fig. 3

- It satisfies the normalization condition

$$\lim_{|z|\to 0}(mz/2)^{1-\lambda_1}\psi_1(x,y)=\frac{1}{\Gamma(\lambda_1)}, \tag{2.16}$$

where the fractional power of z is defined as

$$z^{1-\lambda_1}=e^{(1-\lambda_1)\ln z},\qquad 0<\mathrm{Im}(\ln z)<2\pi.$$

Theorem 2.1 shows that these requirements determine the solution uniquely.

The function $e^{-2\pi i\lambda_0 y/\beta}\psi$ is periodic in the left halfplane and therefore can be expanded there in Fourier series. Substituting the series into Dirac equation, one obtains a general form of the solution for $x<0$,

$$\psi_{x<0}(x,y)=-A\sum_{n\in\mathbb{Z}+\lambda_0}\frac{G(\theta_n)}{m\beta\cosh\theta_n}\,e^{mx\cosh\theta_n+imy\sinh\theta_n}\begin{pmatrix}e^{\theta_n}\\ 1\end{pmatrix}, \tag{2.17}$$

where $\sinh\theta_n=\frac{2\pi}{m\beta}n$ and the factor $1/(m\beta\cosh\theta_n)$ is introduced for further convenience. Analogously, the general form of the solution in the right halfplane is

$$\psi_{x>0}(x,y)=A\sum_{n\in\mathbb{Z}-\tilde{\lambda}}\frac{H(\theta_n)}{m\beta\cosh\theta_n}\,e^{-mx\cosh\theta_n-imy\sinh\theta_n}\begin{pmatrix}-e^{\theta_n}\\ 1\end{pmatrix}. \tag{2.18}$$

Of course, in order for the series (2.17) and (2.18) to converge the functions $G(\theta)$ and $H(\theta)$ have to not grow too rapidly as $\theta\to\pm\infty$. Moreover, we shall assume that $G(\theta)$ and $H(\theta)$ are analytic in the strip $-\frac{\pi}{2}-\delta<\mathrm{Im}\,\theta<\frac{\pi}{2}+\delta$ for some $\delta>0$, so that (2.17) and (2.18) can be represented via contour integrals (see Fig. 3)

$$\psi_{x<0}(x,y)=A\int_{C_-\bigcup C_+}\frac{d\theta}{2\pi}\,\frac{G(\theta)}{1-e^{im\beta\sinh\theta-2\pi i\lambda_0}}\,e^{mx\cosh\theta+imy\sinh\theta}\begin{pmatrix}e^{\theta}\\ 1\end{pmatrix},$$

$$\psi_{x>0}(x,y)=A\int_{C_-\bigcup C_+}\frac{d\theta}{2\pi}\,\frac{H(\theta)}{1-e^{-im\beta\sinh\theta-2\pi i\tilde{\lambda}}}\,e^{-mx\cosh\theta-imy\sinh\theta}\begin{pmatrix}-e^{\theta}\\ 1\end{pmatrix}.$$

If $0 < y < \beta$, the contours C_+ and C_- can be continuously deformed into $\operatorname{Im}\theta = \frac{\pi}{2}$ and $\operatorname{Im}\theta = -\frac{\pi}{2}$, respectively, defining the continuations of $\psi_{x<0}(x, y)$ and $\psi_{x>0}(x, y)$ on the whole strip

$$\psi_{x<0}(x, y) = A \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \left\{ -\frac{G(\theta + i\pi/2)e^{imx\sinh\theta - my\cosh\theta}}{1 - e^{-m\beta\cosh\theta - 2\pi i\lambda_0}} \begin{pmatrix} ie^\theta \\ 1 \end{pmatrix} \right.$$
$$\left. + \frac{G(\theta - i\pi/2)e^{-imx\sinh\theta + my\cosh\theta}}{1 - e^{m\beta\cosh\theta - 2\pi i\lambda_0}} \begin{pmatrix} -ie^\theta \\ 1 \end{pmatrix} \right\},$$
$$\psi_{x>0}(x, y) = A \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \left\{ -\frac{H(\theta + i\pi/2)e^{-imx\sinh\theta + my\cosh\theta}}{1 - e^{m\beta\cosh\theta - 2\pi i\tilde{\lambda}}} \begin{pmatrix} -ie^\theta \\ 1 \end{pmatrix} \right.$$
$$\left. + \frac{H(\theta - i\pi/2)e^{imx\sinh\theta - my\cosh\theta}}{1 - e^{-m\beta\cosh\theta - 2\pi i\tilde{\lambda}}} \begin{pmatrix} ie^\theta \\ 1 \end{pmatrix} \right\}.$$

These continuations coincide if two functional relations for $G(\theta)$ and $H(\theta)$ hold:

$$\frac{G(\theta + i\pi/2)}{H(\theta - i\pi/2)} = -\frac{1 - e^{-m\beta\cosh\theta - 2\pi i\lambda_0}}{1 - e^{-m\beta\cosh\theta - 2\pi i\tilde{\lambda}}}, \tag{2.19}$$
$$\frac{G(\theta - i\pi/2)}{H(\theta + i\pi/2)} = -e^{2\pi i\lambda_1} \frac{1 - e^{-m\beta\cosh\theta + 2\pi i\lambda_0}}{1 - e^{-m\beta\cosh\theta + 2\pi i\tilde{\lambda}}}. \tag{2.20}$$

The relevant solutions of these equations can be found using the following lemma.

Lemma 2.2. *Consider two functions, $f(\theta)$ and $g(\theta)$, that are analytic in the strip $|\operatorname{Im}\theta| < \delta$. If in this strip $|f(\theta)| = O\left(\frac{1}{|\operatorname{Re}\theta|^2}\right)$ and $|g(\theta)| = O(1)$ as $\operatorname{Re}\theta \to \pm\infty$, then the functions*

$$\nu(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \tanh(\theta' - \theta) f(\theta'), \quad \eta(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \operatorname{sech}(\theta' - \theta) g(\theta'), \quad \theta \in \mathbb{R} \tag{2.21}$$

can be analytically continued to the strip $|\operatorname{Im}\theta| < \frac{\pi}{2} + \delta$. Furthermore, if $|\operatorname{Im}\theta| < \delta$, these continuations satisfy the relations

$$\nu\Big(\theta + \frac{i\pi}{2}\Big) - \nu\Big(\theta - \frac{i\pi}{2}\Big) = -if(\theta), \qquad \eta\Big(\theta + \frac{i\pi}{2}\Big) + \eta\Big(\theta - \frac{i\pi}{2}\Big) = g(\theta). \tag{2.22}$$

■ Obviously, the expressions (2.21) for $\nu(\theta)$ and $\eta(\theta)$ are analytic functions in the strip $|\mathrm{Im}\,\theta| < \frac{\pi}{2}$. Their analytic continuations to $|\mathrm{Im}\,\theta| < \frac{\pi}{2} + \delta$ are

$$\nu(\theta)\Big|_{\mathrm{Im}\,\theta=\pm\frac{\pi}{2}} = \mp\frac{i}{2}\, f(\theta \mp \frac{i\pi}{2}) + P\int_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\coth\Big(\theta' - \theta \pm \frac{i\pi}{2}\Big) f(\theta'),$$

$$\eta(\theta)\Big|_{\mathrm{Im}\,\theta=\pm\frac{\pi}{2}} = \frac{1}{2}\, g(\theta \mp \frac{i\pi}{2}) \pm i\, P\int_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\,\mathrm{csch}\Big(\theta' - \theta \pm \frac{i\pi}{2}\Big) g(\theta'),$$

$$\nu(\theta)\Big|_{\frac{\pi}{2}<\mathrm{Im}\,\theta<\frac{\pi}{2}+\delta} = -if\Big(\theta - \frac{i\pi}{2}\Big) + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\tanh(\theta' - \theta) f(\theta'),$$

$$\eta(\theta)\Big|_{\frac{\pi}{2}<\mathrm{Im}\,\theta<\frac{\pi}{2}+\delta} = g\Big(\theta - \frac{i\pi}{2}\Big) + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\,\mathrm{sech}(\theta' - \theta) g(\theta'),$$

$$\nu(\theta)\Big|_{-\frac{\pi}{2}-\delta<\mathrm{Im}\,\theta<-\frac{\pi}{2}} = if\Big(\theta + \frac{i\pi}{2}\Big) + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\tanh(\theta' - \theta) f(\theta'),$$

$$\eta(\theta)\Big|_{-\frac{\pi}{2}-\delta<\mathrm{Im}\,\theta<-\frac{\pi}{2}} = g\Big(\theta + \frac{i\pi}{2}\Big) + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\,\mathrm{sech}(\theta' - \theta) g(\theta'),$$

where the first two integrals are understood in the principal value sense. Then the statement of the lemma follows immediately. □

If we write the functions $G(\theta)$ and $H(\theta)$ in the form

$$\begin{cases} G(\theta) = -\exp\left(\pi i\lambda_1 - \lambda_1\theta + \frac{i}{2}\,\nu(\theta) + \frac{1}{2}\eta(\theta)\right), \\ H(\theta) = \exp\left(-\lambda_1\theta + \frac{i}{2}\,\nu(\theta) - \frac{1}{2}\eta(\theta)\right), \end{cases} \tag{2.23}$$

the functional relations (2.19) and (2.20) reduce to (2.22) with a particular choice of the functions $f(\theta)$ and $g(\theta)$:

$$f(\theta) = \ln\frac{1 - e^{-m\beta\cosh\theta - 2\pi i\lambda_0}}{1 - e^{-m\beta\cosh\theta + 2\pi i\lambda_0}} - \ln\frac{1 - e^{-m\beta\cosh\theta - 2\pi i\tilde{\lambda}}}{1 - e^{-m\beta\cosh\theta + 2\pi i\tilde{\lambda}}}, \tag{2.24}$$

$$g(\theta) = \ln\frac{(1 - e^{-m\beta\cosh\theta + 2\pi i\lambda_0})(1 - e^{-m\beta\cosh\theta - 2\pi i\lambda_0})}{(1 - e^{-m\beta\cosh\theta + 2\pi i\tilde{\lambda}})(1 - e^{-m\beta\cosh\theta - 2\pi i\tilde{\lambda}})}. \tag{2.25}$$

The branches of logarithms in (2.24) and (2.25) are fixed so that for real θ their imaginary parts lie in the interval $(-\pi;\pi)$.

The formulae (2.17), (2.18), (2.21) and (2.23)–(2.25) provide a solution to the Dirac equation on the 1-punctured cylinder with specified branching (one should go back and check that all formal manipulations we have made with functions $G(\theta)$ and $H(\theta)$ indeed can be done). It remains only to verify the normalization condition (2.16).

Let us take, say, the expansion (2.18) and rewrite it using the Poisson summation formula:

$$\psi_{x>0}(x,y) = A\sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\, H(\theta)\, e^{-mx\cosh\theta - im(y+k\beta)\sinh\theta - 2\pi ik\tilde{\lambda}} \begin{pmatrix} -e^{\theta} \\ 1 \end{pmatrix}.$$

The asymptotics of ψ for $|z| \to 0$ is determined by the term with $k = 0$. Since $\lambda_1 > 0$, the main contribution to the corresponding integral is due to large $|\theta|$. Straightforward calculation shows that for $|z| \to 0$,

$$\psi_1(x,y) \sim -\frac{A}{2\pi}\, e^{-\pi i\lambda_1/2 + i\nu_\infty/2}\Gamma(1-\lambda_1)(mz/2)^{\lambda_1 - 1},$$

where

$$\begin{aligned} \nu_\infty &= \lim_{\theta\to+\infty} \nu(\theta) \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta \left(\ln\frac{1-e^{-m\beta\cosh\theta - 2\pi i\lambda_0}}{1-e^{-m\beta\cosh\theta + 2\pi i\lambda_0}} - \ln\frac{1-e^{-m\beta\cosh\theta - 2\pi i\tilde{\lambda}}}{1-e^{-m\beta\cosh\theta + 2\pi i\tilde{\lambda}}} \right). \end{aligned} \tag{2.26}$$

Therefore, the solution we have constructed differs from the element of the canonical basis only by a constant factor which can be set to unity by the appropriate choice of A. Summarizing all these results, we obtain

Theorem 2.3. *The element of canonical basis on the cylinder with one branchpoint is given by the following expressions:*

$$\mathbf{w}(x,y) = A\sum_{n\in\mathbb{Z}+\lambda_0} \frac{e^{\pi i\lambda_1 + \frac{i}{2}\nu(\theta_n) + \frac{1}{2}\eta(\theta_n)}}{m\beta\cosh\theta_n}\, e^{-\lambda_1\theta_n + mx\cosh\theta_n + imy\sinh\theta_n} \begin{pmatrix} e^{\theta_n} \\ 1 \end{pmatrix} \quad \textit{for } x<0,$$

$$\mathbf{w}(x,y) = A\sum_{n\in\mathbb{Z}-\tilde{\lambda}} \frac{e^{\frac{i}{2}\nu(\theta_n) - \frac{1}{2}\eta(\theta_n)}}{m\beta\cosh\theta_n}\, e^{-\lambda_1\theta_n - mx\cosh\theta_n - imy\sinh\theta_n} \begin{pmatrix} -e^{\theta_n} \\ 1 \end{pmatrix} \quad \textit{for } x>0,$$

where the functions $\nu(\theta)$, $\eta(\theta)$ are determined from

$$\nu(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \tanh(\theta'-\theta) \left(\ln\frac{1-e^{-m\beta\cosh\theta' - 2\pi i\lambda_0}}{1-e^{-m\beta\cosh\theta' + 2\pi i\lambda_0}} - \ln\frac{1-e^{-m\beta\cosh\theta' - 2\pi i\tilde{\lambda}}}{1-e^{-m\beta\cosh\theta' + 2\pi i\tilde{\lambda}}} \right),$$

$$\eta(\theta) = \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\, \mathrm{sech}\,(\theta'-\theta) \ln\frac{(1-e^{-m\beta\cosh\theta' + 2\pi i\lambda_0})(1-e^{-m\beta\cosh\theta' - 2\pi i\lambda_0})}{(1-e^{-m\beta\cosh\theta' + 2\pi i\tilde{\lambda}})(1-e^{-m\beta\cosh\theta' - 2\pi i\tilde{\lambda}})},$$

and normalization constant $A = -2\sin\pi\lambda_1\, e^{-i\nu_\infty/2}$.

3. Green Function for the Dirac Operator

3.1. Green function for $n = 0$. Let us calculate the Green function on the cylinder without branchpoints. The domain of the Dirac operator in this case consists of quasi-periodic functions $\psi(x, y+\beta) = e^{2\pi i\lambda_0}\psi(x, y)$ that are square integrable in the strip $S = \{(x, y) : 0 < y < \beta\}$. After Fourier transformation

$$\hat{\psi}(\xi_x, \xi_y) = \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} dx \int\limits_{0}^{\beta} dy\ \psi(x, y)\, e^{-i(x\xi_x + y\xi_y)}, \qquad \xi_x \in \mathbb{R},\ \xi_y \in \frac{2\pi}{\beta}(\mathbb{Z} + \lambda_0)$$

the Dirac operator and its inverse are represented by matrices

$$D = \frac{1}{2}\begin{pmatrix} m & -i\bar{\xi} \\ -i\xi & m \end{pmatrix}, \qquad D^{-1} = \frac{2}{m^2 + |\xi|^2}\begin{pmatrix} m & i\bar{\xi} \\ i\xi & m \end{pmatrix},$$

where $\xi = \xi_x + i\xi_y$, $\bar{\xi} = \xi_x - \xi_y$. Since the inverse transformation is given by

$$\psi(x, y) = \frac{2\pi}{\beta}\sum_{\xi_y}\int\limits_{-\infty}^{\infty} d\xi_x\ \hat{\psi}(\xi_x, \xi_y)\, e^{i(x\xi_x + y\xi_y)},$$

one obtains the following formula for the Green function $G_0(x - x', y - y')$:

$$G_0(x, y) = \frac{1}{\pi\beta}\sum_{\xi_y}\int\limits_{-\infty}^{\infty} d\xi_x \begin{pmatrix} m & i\bar{\xi} \\ i\xi & m \end{pmatrix}\frac{e^{i(x\xi_x + y\xi_y)}}{m^2 + |\xi|^2}. \tag{3.1}$$

Another two representations of the Green function will be useful for us. Choose, for example, $x > 0$ and calculate the integrals in (3.1):

$$G_0(x, y) = \sum_{n\in\mathbb{Z}-\lambda_0}\frac{e^{-mx\cosh\theta_n - imy\sinh\theta_n}}{\beta\cosh\theta_n}\begin{pmatrix} 1 & -e^{\theta_n} \\ -e^{-\theta_n} & 1 \end{pmatrix} \tag{3.2}$$

$$= m\int_{C_-\cup C_+}\frac{d\theta}{2\pi}\frac{e^{-mx\cosh\theta - imy\sinh\theta}}{1 - e^{-im\beta\sinh\theta - 2\pi i\lambda_0}}\begin{pmatrix} 1 & -e^{\theta} \\ -e^{-\theta} & 1 \end{pmatrix}. \tag{3.3}$$

Analogously, for $x < 0$ one obtains

$$G_0(x, y) = \sum_{n\in\mathbb{Z}+\lambda_0}\frac{e^{mx\cosh\theta_n + imy\sinh\theta_n}}{\beta\cosh\theta_n}\begin{pmatrix} 1 & e^{\theta_n} \\ e^{-\theta_n} & 1 \end{pmatrix} \tag{3.4}$$

$$= -m\int_{C_-\cup C_+}\frac{d\theta}{2\pi}\frac{e^{mx\cosh\theta + imy\sinh\theta}}{1 - e^{im\beta\sinh\theta - 2\pi i\lambda_0}}\begin{pmatrix} 1 & e^{\theta} \\ e^{-\theta} & 1 \end{pmatrix}. \tag{3.5}$$

In what follows, we shall also need the asymptotics of these expressions as $x, y \to 0$. To find it, let us rewrite (3.1) using the Poisson formula:

$$G_0(x, y) = \frac{1}{2\pi^2}\sum_{k\in\mathbb{Z}}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} d\xi_x d\xi_y \begin{pmatrix} m & i\bar{\xi} \\ i\xi & m \end{pmatrix}\frac{e^{i(x\xi_x + y\xi_y) + ik(\beta\xi_y - 2\pi\lambda_0)}}{m^2 + |\xi|^2}. \tag{3.6}$$

The leading term of asymptotics is determined by the integral corresponding to $k = 0$, representing the Green function on the plane. A little calculation shows that as $|z| \to 0$,

$$G_0(x, y) \sim -\frac{m}{\pi}\begin{pmatrix} \ln|z| & 1/z \\ 1/\bar{z} & \ln|z| \end{pmatrix}.$$

It is very convenient to write the Green function as

$$G_0(z) = 2G(z)J, \qquad J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

since the rows of $G(z)$ satisfy the Dirac equation (and not its adjoint, as the rows of $G_0(z)$ do).

3.2. General properties of Green function. The domain $\mathcal{D}^{a,\lambda}$ of the Dirac operator $D^{a,\lambda}$ is chosen to consist of functions ψ that have monodromies $e^{2\pi i\lambda_\nu}$ ($\nu = 0, \ldots, n$) and are integrable at $|x| \to \infty$. We also require (see (2.6))

$$\begin{pmatrix} (z-a_\nu)^{-\lambda_\nu} & 0 \\ 0 & (\bar{z}-\bar{a}_\nu)^{\lambda_\nu} \end{pmatrix}\psi[a_\nu] \in H^1[a_\nu], \qquad \nu = 1, \ldots, n, \tag{3.7}$$

where $H^1[a_\nu]$ denotes the space of functions that are square integrable in the neighborhood of $\{a_\nu\}$ together with their first derivatives. In the previous section we have shown that the Dirac equation $D^{a,\lambda}\psi = 0$ has no solutions in $\mathcal{D}^{a,\lambda}$. Thus "naively" we can think of $D^{a,\lambda}$ as being an invertible operator. The kernel of the inverse is called the Green function $G^{a,\lambda}$. More precisely, the solution of

$$D^{a,\lambda}\psi = \varphi,$$

is assumed to have the form

$$\psi(z) = \int_{\mathcal{C}\backslash b} G^{a,\lambda}(z, z')J\,\varphi(z')\,i dz' \wedge d\bar{z}'. \tag{3.8}$$

Then one can try to determine the Green function by the following requirements:

- The columns of $G^{a,\lambda}(z, z')$ must satisfy Dirac equation $D_z G^{a,\lambda}_{\cdot,j}(z, z') = 0$ for all $z \in \mathcal{C}\backslash(b \cup \{z'\})$; they are square integrable functions at $|x| \to \infty$ that have monodromy $e^{2\pi i\lambda_\nu}$ ($\nu = 0, \ldots, n$) and singular behaviour (3.7) at each singularity. Therefore,

$$G^{a,\lambda}_{\cdot,j}(z, z')[a_\nu] = \sum_{k>0}\left\{a^{(\nu)}_{k,j}(z')w_{k+\lambda_\nu}[a_\nu] + b^{(\nu)}_{k,j}(z')w^*_{k-\lambda_\nu}[a_\nu]\right\}. \tag{3.9}$$

- The integral operator with the kernel $D^{a,\lambda}_z G^{a,\lambda}(z, z')$ has to "cut out" the values of the function $\varphi(z)$. Therefore, the singular behaviour of $G^{a,\lambda}(z, z')$ for $z \to z'$ must coincide with that of Green function for the Dirac operator on the cylinder without branchpoints,

$$G^{a,\lambda}(z, z') - G(z, z') \in C^1(z \to z'). \tag{3.10}$$

Remark. Suppose the function defined by these conditions exists. Then it is unique, since the columns of the difference of two such functions are obviously in $\widetilde{\mathbf{W}}^{a,\lambda}$.

Next we determine how $G^{a,\lambda}(z,z')$ depends on the second argument. To do this, let us define a matrix $F^{a,\lambda}(z,z')$ satisfying the following conditions:

- The rows of $F^{a,\lambda}(z,z')$ satisfy Dirac equation $D_{z'}G^{a,\lambda}_{j,\cdot}(z,z') = 0$ for all $z' \in \mathcal{C}\backslash(b \cup \{z\})$; they are square integrable functions at $|x| \to \infty$, have the inverse monodromy $e^{-2\pi i\lambda_\nu}$ ($\nu = 0, \ldots, n$) and corresponding singular behaviour (3.7) at the branching points:

$$F^{a,\lambda}_{j,\cdot}(z,z')[a_\nu] = \sum_{k>0}\left\{\alpha^{(\nu)}_{k,j}(z)w_{k-\lambda_\nu}[a_\nu] + \beta^{(\nu)}_{k,j}(z)w^{*}_{k+\lambda_\nu}[a_\nu]\right\}. \qquad (3.11)$$

- The singular behaviour of $F^{a,\lambda}(z,z')$ for $z' \to z$ coincides with the asymptotics of "unperturbed" Green function,

$$F^{a,\lambda}(z,z') - G(z,z') \in C^1(z' \to z). \qquad (3.12)$$

Skipping the proof of existence of $G^{a,\lambda}(z,z')$ (or $F^{a,\lambda}(z,z')$) we shall now prove the following

Theorem 3.1. $G^{a,\lambda}(z,z') = F^{a,\lambda}(z,z')$.

■ First we note an auxiliary relation: if $f(z)$ and $g(z)$ are smooth functions on some open set $U \subset \mathcal{C}$, then

$$\begin{aligned}\{Df \cdot Jg - f \cdot JDg\}\, dz \wedge d\bar{z} &= \left\{\partial_z(f_2 g_2) - \partial_{\bar{z}}(f_1 g_1)\right\} dz \wedge d\bar{z} \\ &= d(f_1 g_1 dz + f_2 g_2 d\bar{z}).\end{aligned} \qquad (3.13)$$

Now choose two distinct points $x, y \notin b$ and

$$\begin{cases} f(z) = F^{a,\lambda}_{i,\cdot}(x,z), \\ g(z) = G^{a,\lambda}_{\cdot,j}(z,y), \end{cases}$$

with U being the complement to the union of the disks $\left(\bigcup_{\nu=1}^{n} D_\varepsilon(a_\nu)\right) \cup D_\varepsilon(x) \cup D_\varepsilon(y)$ and two strips: $S'_\varepsilon = \{(x,y) : 0 \le y < \varepsilon\}$ and $S''_\varepsilon = \{(x,y) : \beta - \varepsilon < y \le \beta\}$. Integrating (3.13) over this set and using Stokes theorem, one obtains

$$\begin{aligned}0 = \sum_{\nu=1}^{n} \oint\limits_{\partial D_\varepsilon(a_\nu)} (f_1 g_1 dz + f_2 g_2 d\bar{z}) + \oint\limits_{\partial D_\varepsilon(x) \cup \partial D_\varepsilon(y)} (f_1 g_1 dz + f_2 g_2 d\bar{z}) \\ + \oint\limits_{\partial S'_\varepsilon \cup \partial S''_\varepsilon} (f_1 g_1 dz + f_2 g_2 d\bar{z}).\end{aligned}$$

The expression under the integrals is single-valued on $\mathcal{C}\backslash a$, so in the limit $\varepsilon \to 0$ the last integral cancels out. The integrals over $\partial D_\varepsilon(a_\nu)$ also vanish. One can easily check this, substituting instead of $G^{a,\lambda}_{\cdot,j}(z,y)$ and $F^{a,\lambda}_{i,\cdot}(x,z)$ their local expansions (3.9) and (3.11). It remains to calculate

$$\lim_{\varepsilon\to 0}\left\{\oint\limits_{\partial D_\varepsilon(x)} F^{a,\lambda}_{i,1}(x,z)G^{a,\lambda}_{1,j}(z,y)dz + \oint\limits_{\partial D_\varepsilon(x)} F^{a,\lambda}_{i,2}(x,z)G^{a,\lambda}_{2,j}(z,y)d\bar{z}\right\}$$
$$= \frac{i}{2\pi}\left\{-2\pi i\delta_{i1}G^{a,\lambda}_{1,j}(x,y) - 2\pi i\delta_{i2}G^{a,\lambda}_{2,j}(x,y)\right\} = G^{a,\lambda}_{i,j}(x,y),$$

and, similarly,

$$\lim_{\varepsilon\to 0}\left\{\oint\limits_{\partial D_\varepsilon(y)} F^{a,\lambda}_{i,1}(x,z)G^{a,\lambda}_{1,j}(z,y)dz + \oint\limits_{\partial D_\varepsilon(y)} F^{a,\lambda}_{i,2}(x,z)G^{a,\lambda}_{2,j}(z,y)d\bar{z}\right\} = -F^{a,\lambda}_{i,j}(x,y).$$

Finally we get $G^{a,\lambda}_{i,j}(x,y) = F^{a,\lambda}_{i,j}(x,y)$. □

Remark. Since the columns and rows of $G^{a,\lambda}(z,z')$ satisfy Dirac equation in z and z' respectively, so do also the derivatives of the Green function $\partial_{a_\mu}G^{a,\lambda}(z,z')$ and $\partial_{\bar{a}_\mu}G^{a,\lambda}(z,z')$. These derivatives are not singular at $z\to z'$. However, their local expansions are more singular than (3.9) or (3.11); from the relations

$$\begin{cases}\partial_z w_l = \frac{m}{2}w_{l-1},\\ \partial_{\bar{z}} w_l = \frac{m}{2}w_{l+1},\end{cases}\qquad \begin{cases}\partial_z w^*_l = \frac{m}{2}w^*_{l+1},\\ \partial_{\bar{z}} w^*_l = \frac{m}{2}w^*_{l-1},\end{cases}$$

it follows that

$$\partial_{a_\mu}G^{a,\lambda}_{j,\cdot}(z,z')[a_\nu] = -\frac{m}{2}\delta_{\mu\nu}\alpha^{(\nu)}_{1/2,j}(z)w_{-1/2-\lambda_\nu} + \sum_{k>0}\left\{\gamma^{(\nu)}_{k,j}(z)w_{k-\lambda_\nu} + \tilde{\gamma}^{(\nu)}_{k,j}(z)w^*_{k+\lambda_\nu}\right\}, \tag{3.14}$$

$$\partial_{\bar{a}_\mu}G^{a,\lambda}_{j,\cdot}(z,z')[a_\nu] = -\frac{m}{2}\delta_{\mu\nu}\beta^{(\nu)}_{1/2,j}(z)w^*_{-1/2+\lambda_\nu} + \sum_{k>0}\left\{\eta^{(\nu)}_{k,j}(z)w_{k-\lambda_\nu} + \tilde{\eta}^{(\nu)}_{k,j}(z)w^*_{k+\lambda_\nu}\right\}. \tag{3.15}$$

Therefore, if we somehow determine the coefficients $\alpha^{(\nu)}_{1/2,j}(z)$, $\beta^{(\nu)}_{1/2,j}(z)$ and find the solution of Dirac equation with the same "extra" singular behaviour, it will coincide with the corresponding derivative of Green function.

Let us consider a multivalued solution $f(z)$ to the Dirac equation in the strip $S = \{(x,y) : 0 < y < \beta\}$, which is square integrable at $|x|\to\infty$ and has the following local expansions about the branchpoints a_ν (there are no other singularities!)

$$f(z')[a_\nu] = \sum_k\left\{a^{(\nu)}_k w_{k-\lambda_\nu} + b^{(\nu)}_k w^*_{k+\lambda_\nu}\right\}.$$

In addition, only a finite number of coefficients $a^{(\nu)}_k$, $b^{(\nu)}_k$ with $k<0$ is allowed to have non-zero values. Lower and upper continuations of $f(z)$ across the branchcut d_ν

($\nu = 0, \dots, n$) are supposed to differ by monodromy multiplier $\exp\big(2\pi i \sum_{k=0}^{\nu} \lambda_k\big)$, so that the product $G^{a,\lambda}_{j,\cdot}(z,z')\overline{f(z')}$, as the function of z', is single-valued on $\mathcal{C}\backslash a$. Using (2.9) and Stokes theorem, one obtains

$$\frac{m}{2} \int\limits_{\mathcal{C}\backslash \bigcup_\nu D_\varepsilon(a_\nu)} G^{a,\lambda}_{j,\cdot}(z,z')\overline{f(z')} i dz' \wedge d\bar{z}' \tag{3.16}$$

$$= -\sum_\nu \oint\limits_{\partial D_\varepsilon(a_\nu)} G^{a,\lambda}_{j,2}(z,z')\overline{f_1(z')} i d\bar{z}' - \oint\limits_{\partial D_\varepsilon(z)} G^{a,\lambda}_{j,2}(z,z')\overline{f_1(z')} i d\bar{z}' \tag{3.17}$$

$$= \sum_\nu \oint\limits_{\partial D_\varepsilon(a_\nu)} G^{a,\lambda}_{j,1}(z,z')\overline{f_2(z')} i dz' + \oint\limits_{\partial D_\varepsilon(z)} G^{a,\lambda}_{j,1}(z,z')\overline{f_2(z')} i dz'. \tag{3.18}$$

The surface integral (3.16) does not converge for $\varepsilon \to 0$. However, one can still compare the asymptotics of (3.17) and (3.18) for $\varepsilon \to 0$:

$$\lim_{\varepsilon\to 0} (3.17) = -\frac{4}{m} \sum_\nu \sum_{k>0} (-1)^{k+1/2} \sin\pi\lambda_\nu\, \beta^{(\nu)}_{k,j}(z)\overline{a^{(\nu)}_{-k}} - i\delta_{j2}\overline{f_1(z)},$$

$$\lim_{\varepsilon\to 0} (3.18) = \frac{4}{m} \sum_\nu \sum_{k>0} (-1)^{k+1/2} \sin\pi\lambda_\nu\, \alpha^{(\nu)}_{k,j}(z)\overline{b^{(\nu)}_{-k}} + i\delta_{j1}\overline{f_2(z)}.$$

Finally we get

$$\sum_\nu \sum_{k>0} (-1)^{k+1/2} \sin\pi\lambda_\nu \left\{\beta^{(\nu)}_{k,j}(z)\overline{a^{(\nu)}_{-k}} + \alpha^{(\nu)}_{k,j}(z)\overline{b^{(\nu)}_{-k}}\right\} = -\frac{im}{4}\big(\delta_{j2}\overline{f_1(z)} + \delta_{j1}\overline{f_2(z)}\big). \tag{3.19}$$

Now we shall make a special choice of $f(z')$ to find the lowest coefficients of the Green function expansions. Analogously to the previous section (see (2.12)), let us introduce n special multivalued solutions to Dirac equation, that are integrable at $|x| \to \infty$ and have local expansions

$$\begin{aligned}\widetilde{\mathbf{w}}_\mu(\lambda)[a_\nu] = {}& \delta_{\mu\nu} w_{-1/2+\lambda_\nu}[a_\nu] \\ &+ \sum_{k>0} \left\{a^{(\nu)}_k(\widetilde{\mathbf{w}}_\mu(\lambda)) w_{k+\lambda_\nu}[a_\nu] + b^{(\nu)}_k(\widetilde{\mathbf{w}}_\mu(\lambda)) w^*_{k-\lambda_\nu}[a_\nu]\right\}.\end{aligned} \tag{3.20}$$

The existence and uniqueness of such solutions follow from the existence of canonical basis; two bases coincide if all $\lambda_\nu > 0$.

After the substitution $f(z) = \widetilde{\mathbf{w}}_\mu(-\lambda)$ in (3.19) one obtains

$$\begin{pmatrix} \beta^{(\mu)}_{1/2,1}(z) \\ \beta^{(\mu)}_{1/2,2}(z) \end{pmatrix} = \frac{im}{4\sin\pi\lambda_\mu} \begin{pmatrix} \overline{\widetilde{\mathbf{w}}_{\mu 2}(z,-\lambda)} \\ \overline{\widetilde{\mathbf{w}}_{\mu 1}(z,-\lambda)} \end{pmatrix} = \frac{im}{4\sin\pi\lambda_\mu} \widetilde{\mathbf{w}}^*_\mu(z,-\lambda). \tag{3.21}$$

On the other hand, the substitution $f(z) = \widetilde{\mathbf{w}}^*_\mu(z,\lambda)$ in (3.19) gives a formula for $\alpha^{(\mu)}_{1/2,j}(z)$,

$$\begin{pmatrix} \alpha^{(\mu)}_{1/2,1}(z) \\ \alpha^{(\mu)}_{1/2,2}(z) \end{pmatrix} = \frac{im}{4\sin\pi\lambda_\mu}\,\widetilde{\mathbf{w}}_\mu(z,\lambda). \tag{3.22}$$

Taking into account the earlier remarks (the formulae (3.14) and (3.15)), we obtain an expression for the derivatives of the Green function in terms of the solutions (3.20):

$$\partial_{a_j} G^{a,\lambda}(z,z') = -\frac{im^2}{8\sin\pi\lambda_j}\,\widetilde{\mathbf{w}}_j(z,\lambda)\otimes\widetilde{\mathbf{w}}_j(z',-\lambda), \tag{3.23}$$

$$\partial_{\bar{a}_j} G^{a,\lambda}(z,z') = -\frac{im^2}{8\sin\pi\lambda_j}\,\widetilde{\mathbf{w}}^*_j(z,-\lambda)\otimes\widetilde{\mathbf{w}}^*_j(z',\lambda). \tag{3.24}$$

3.3. One-point Green function. The formulae (3.23) and (3.24) can be used to calculate the Green function $G^{a,\lambda}$ on the cylinder with one branching point $\{a\}$. Notice that in the case of a single puncture the solutions (3.20) are easily expressed via the element of canonical basis. If we suppose for definiteness that $0 < \lambda_1 < \frac{1}{2}$ then $\tilde{\mathbf{w}}(z,\lambda) = \mathbf{w}(z,\lambda)$. Differentiating the local expansions, one can also verify that

$$\tilde{\mathbf{w}}(z,-\lambda) = \frac{2}{m}\partial_z\mathbf{w}(z,1-\lambda).$$

Let us determine $G^{a,\lambda}(z,z')$ if both $z,z' \in \mathcal{C}\backslash b$ are in the left half-strip: $\operatorname{Re} z, \operatorname{Re} z' < \operatorname{Re} a$. Using the results of the previous section, one can write

$$\tilde{\mathbf{w}}(z,\lambda) = -A(\lambda)\sum_{l\in\mathbb{Z}+\lambda_0}\frac{G(\theta_l;\lambda)e^{\frac{m}{2}(z-a)e^{\theta_l}+\frac{m}{2}(\bar{z}-\bar{a})e^{-\theta_l}}}{m\beta\cosh\theta_l}\begin{pmatrix} e^{\theta_l} \\ 1 \end{pmatrix}, \tag{3.25}$$

$$\tilde{\mathbf{w}}(z,-\lambda) = -A(1-\lambda)\sum_{n\in\mathbb{Z}-\lambda_0}\frac{G(\theta_n;1-\lambda)e^{\theta_n+\frac{m}{2}(z-a)e^{\theta_n}+\frac{m}{2}(\bar{z}-\bar{a})e^{-\theta_n}}}{m\beta\cosh\theta_n}\begin{pmatrix} e^{\theta_n} \\ 1 \end{pmatrix}. \tag{3.26}$$

Note the simple relations

$$\begin{aligned} A(\lambda)A(1-\lambda) &= 4\sin^2\pi\lambda_1, \\ \nu(\theta;\lambda) &= -\nu(\theta;1-\lambda) = -\nu(\theta;-\lambda), \\ \eta(\theta;\lambda) &= \eta(\theta;1-\lambda) = \eta(\theta;-\lambda). \end{aligned} \tag{3.27}$$

If we substitute (3.25) and (3.26) into Eqs. (3.23), (3.24) and then integrate them, we obtain the following formula for Green function $G^{a,\lambda}(z,z')$:

$$G^{a,\lambda}(z,z') = i\sin\pi\lambda_1\sum_{l\in\mathbb{Z}+\lambda_0}\sum_{n\in\mathbb{Z}-\lambda_0}\frac{G(\theta_l;\lambda)G(\theta_n;1-\lambda)e^{\theta_n}}{m\beta^2\cosh\theta_l\cosh\theta_n}$$
$$\times\frac{e^{\frac{m}{2}\{(z-a)e^{\theta_l}+(\bar{z}-\bar{a})e^{-\theta_l}+(z'-a)e^{\theta_n}+(\bar{z}'-\bar{a})e^{-\theta_n}\}}}{e^{\theta_l}+e^{\theta_n}}\begin{pmatrix} e^{\theta_l+\theta_n} & e^{\theta_l} \\ e^{\theta_n} & 1 \end{pmatrix} + C(z,z'). \tag{3.28}$$

The function $C(z, z')$ does not depend on the position of the branching point and has to be determined. We remark that the double sum in the RHS of the last relation converges even if $z = z'$. This gives a hint that $C(z, z')$ should be equal to "unperturbed" Green function $G(z - z'; \lambda_0)$. Assuming this is indeed the case, let us fix $a = 0$ and rewrite (3.28) via contour integrals

$$G^{0,\lambda}(z, z') = im \sin \pi \lambda_1 \int\limits_{C_- \cup C_+} \frac{d\theta}{2\pi} \int\limits_{C_- \cup C_+} \frac{d\theta'}{2\pi} \frac{G(\theta; \lambda) G(\theta'; 1-\lambda) e^{\theta'}}{(1 - e^{im\beta \sinh \theta - 2\pi i \lambda_0})(1 - e^{im\beta \sinh \theta' + 2\pi i \lambda_0})}$$
$$\times \frac{e^{\frac{m}{2}\{ z e^{\theta} + \bar{z} e^{-\theta} + z' e^{\theta'} + \bar{z}' e^{-\theta'} \}}}{e^{\theta} + e^{\theta'}} \begin{pmatrix} e^{\theta+\theta'} & e^{\theta} \\ e^{\theta'} & 1 \end{pmatrix} + G(z - z'; \lambda_0). \tag{3.29}$$

Analogously, if $\mathrm{Re}\, z, \mathrm{Re}\, z' > 0$, the Green function $G^{0,\lambda}(z, z')$ is assumed to have the following form:

$$G^{0,\lambda}(z, z') = im \sin \pi \lambda_1 \int\limits_{C_- \cup C_+} \frac{d\theta}{2\pi} \int\limits_{C_- \cup C_+} \frac{d\theta'}{2\pi} \frac{H(\theta; \lambda) H(\theta'; 1-\lambda) e^{\theta'}}{(1 - e^{-im\beta \sinh \theta - 2\pi i \tilde{\lambda}})(1 - e^{-im\beta \sinh \theta' + 2\pi i \tilde{\lambda}})}$$
$$\times \frac{e^{-\frac{m}{2}\{ z e^{\theta} + \bar{z} e^{-\theta} + z' e^{\theta'} + \bar{z}' e^{-\theta'} \}}}{e^{\theta} + e^{\theta'}} \begin{pmatrix} e^{\theta+\theta'} & -e^{\theta} \\ -e^{\theta'} & 1 \end{pmatrix} + G(z - z'; \tilde{\lambda}). \tag{3.30}$$

In order to prove that the formulae (3.29) and (3.30) indeed represent the Green function, we shall construct their continuations to arbitrary values $z, z' \in \mathcal{C} \backslash b$, and show that these continuations coincide with each other.

Let us start, say, from the representation (3.29). At the first stage, we construct its continuation to arbitrary values of z only. This can be done by shifting the contours C_- and C_+ in the integral over θ to $\mathrm{Im}\, \theta = -\frac{\pi}{2}$, $\mathrm{Im}\, \theta = \frac{\pi}{2}$ respectively. After this procedure one obtains

$$\begin{aligned} & G^{0,\lambda}(z, z') - G(z - z'; \lambda_0) \\ & \quad = im \sin \pi \lambda_1 \int\limits_{-\infty}^{\infty} \frac{d\theta}{2\pi} \int\limits_{C_- \cup C_+} \frac{d\theta'}{2\pi} \frac{G(\theta'; 1-\lambda) e^{\theta' + mx' \cosh \theta' + imy' \sinh \theta'}}{1 - e^{im\beta \sinh \theta' + 2\pi i \lambda_0}} \\ & \qquad \times \left\{ \frac{G(\theta - i\pi/2; \lambda) e^{-imx \sinh \theta + my \cosh \theta}}{(1 - e^{m\beta \cosh \theta - 2\pi i \lambda_0})(-ie^{\theta} + e^{\theta'})} \begin{pmatrix} -ie^{\theta+\theta'} & -ie^{\theta} \\ e^{\theta'} & 1 \end{pmatrix} \right. \\ & \qquad \left. - \frac{G(\theta + i\pi/2; \lambda) e^{imx \sinh \theta - my \cosh \theta}}{(1 - e^{-m\beta \cosh \theta - 2\pi i \lambda_0})(ie^{\theta} + e^{\theta'})} \begin{pmatrix} ie^{\theta+\theta'} & ie^{\theta} \\ e^{\theta'} & 1 \end{pmatrix} \right\}. \end{aligned}$$

We cannot do the same thing the second time, since the function standing in the integral over θ' has the poles $\theta' = \theta \pm \frac{i\pi}{2}$. However, one can shift C_- and C_+ to the contours $C_-^{\varepsilon,\theta}$ and $C_+^{\varepsilon,\theta}$ shown in Fig. 4, and then let $\varepsilon \to 0$. The continuation of (3.29) to all $z, z' \in \mathcal{C} \backslash b$ is then

$$
\begin{aligned}
G^{0,\lambda}(z,z') = {} & im \sin \pi\lambda_1 \sum_{\sigma=\pm 1} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \\
& \times \frac{G(\theta + \frac{i\sigma\pi}{2};\lambda) G(\theta' + \frac{i\sigma\pi}{2}; 1-\lambda) e^{\theta'}}{(1-e^{-\sigma m\beta\cosh\theta - 2\pi i\lambda_0})(1-e^{-\sigma m\beta\cosh\theta' + 2\pi i\lambda_0})} \\
& \times \frac{e^{\sigma m(ix\sinh\theta - y\cosh\theta + ix'\sinh\theta' - y'\cosh\theta')}}{e^{\theta}+e^{\theta'}} \begin{pmatrix} -e^{\theta+\theta'} & \sigma i e^{\theta} \\ \sigma i e^{\theta'} & 1 \end{pmatrix} \\
& - im \sin\pi\lambda_1 \sum_{\sigma=\pm 1} \int_{-\infty}^{\infty} \\
& \times \frac{d\theta}{2\pi} \, P\!\int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \, \frac{G(\theta + \frac{i\sigma\pi}{2};\lambda) G(\theta' - \frac{i\sigma\pi}{2}; 1-\lambda) e^{\theta'}}{(1-e^{-\sigma m\beta\cosh\theta - 2\pi i\lambda_0})(1-e^{\sigma m\beta\cosh\theta' + 2\pi i\lambda_0})} \\
& \times \frac{e^{\sigma m(ix\sinh\theta - y\cosh\theta - ix'\sinh\theta' + y'\cosh\theta')}}{e^{\theta'}-e^{\theta}} \begin{pmatrix} e^{\theta+\theta'} & \sigma i e^{\theta} \\ -\sigma i e^{\theta'} & 1 \end{pmatrix} \\
& + \frac{1}{2}\left\{ G(z-z';\lambda_0) + G(z-z';\tilde{\lambda}) \right\}. \qquad (3.31)
\end{aligned}
$$

The last two terms are "pole contributions" that can be calculated using (3.27) and contour representations of the Green function on the cylinder without branchpoints.

If we perform the same manipulations with the representation (3.30) in the right half-strip, the final expression will coincide with (3.31) due to the relations (2.19) and (2.20) satisfied by the functions $G(\theta)$ and $H(\theta)$. Thus the formulae (3.29)–(3.31) indeed define the Green function $G^{0,\lambda}(z,z')$.

4. Tau Functions

In this section we study the spaces of boundary values of some local solutions to the Dirac equation. These spaces can be embedded into an infinite-dimensional grassmannian.

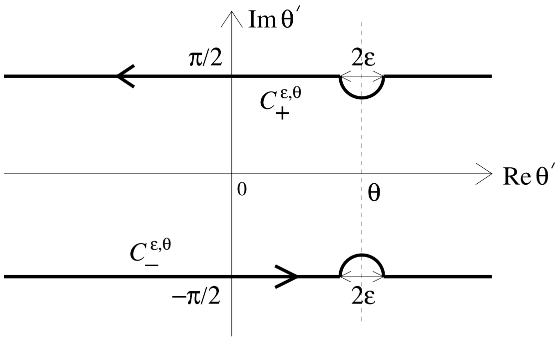

Fig. 4

The τ-functions are determined via the trivialization of the det*-bundle over this grassmannian.

4.1. Subspaces $W_{int}(a)$ and $W_{ext}(a)$. Let us consider a circle $L_{x_0} = \{(x, y) \in \mathcal{C} : x = x_0\}$, and denote by $H_\lambda^{1/2}(L_{x_0})$ the space of $\mathbb{C}^2$-valued quasiperiodic functions on L_{x_0}. Namely, if $g \in H_\lambda^{1/2}(L_{x_0})$, then $g(y + \beta) = e^{2\pi i \lambda} g(y)$. After Fourier transform the function g can be written as

$$g(y) = \frac{2\pi}{\beta} \sum_{n \in \mathbb{Z}+\lambda} \hat{g}(\theta_n) e^{imy \sinh \theta_n}, \qquad \sinh \theta_n = \frac{2\pi}{m\beta} n.$$

Let us introduce two operators, Q_+ and Q_-, acting on $H_\lambda^{1/2}(L_{x_0})$ in the following way:

$$Q_\pm g(y) = \frac{2\pi}{\beta} \sum_{n \in \mathbb{Z}+\lambda} Q_\pm(\theta_n) \hat{g}(\theta_n) e^{imy \sinh \theta_n},$$
$$Q_+(\theta) = \frac{1}{2\cosh\theta} \begin{pmatrix} e^{\theta} & 1 \\ 1 & e^{-\theta} \end{pmatrix}, \qquad Q_-(\theta) = \frac{1}{2\cosh\theta} \begin{pmatrix} e^{-\theta} & -1 \\ -1 & e^{\theta} \end{pmatrix}.$$

These operators have the properties of projectors,

$$Q_+ + Q_- = \mathbf{1}, \qquad Q_+^2 = Q_+, \qquad Q_-^2 = Q_-,$$

and thus define the splitting $H_\lambda^{1/2}(L_{x_0}) = H_\lambda^+ \oplus H_\lambda^-$, with $H_\lambda^\pm = Q_\pm H_\lambda^{1/2}(L_{x_0})$. One may easily verify that

$$\sum_{n \in \mathbb{Z}+\lambda} \| Q_\pm(\theta_n) \hat{g}(\theta_n) \|^2 \cosh\theta_n = \frac{1}{2} \sum_{n \in \mathbb{Z}+\lambda} |g_\pm(\theta_n)|^2,$$

where

$$\begin{pmatrix} g_+(\theta_n) \\ g_-(\theta_n) \end{pmatrix} = \begin{pmatrix} v_n^{1/2} & v_n^{-1/2} \\ -v_n^{-1/2} & v_n^{1/2} \end{pmatrix} \begin{pmatrix} \hat{g}_1(\theta_n) \\ \hat{g}_2(\theta_n) \end{pmatrix}, \qquad v_n \equiv v(\theta_n) = e^{\theta_n}.$$

Therefore, the function g is expressed through its polarization components $g_\pm(\theta_n)$ as

$$g(y) = \frac{2\pi}{\beta} \sum_{n \in \mathbb{Z}+\lambda} \frac{e^{imy \sinh\theta_n}}{2\cosh\theta_n} \begin{pmatrix} v_n^{1/2} & -v_n^{-1/2} \\ v_n^{-1/2} & v_n^{1/2} \end{pmatrix} \begin{pmatrix} g_+(\theta_n) \\ g_-(\theta_n) \end{pmatrix}. \tag{4.1}$$

Let us now show that the elements of H_λ^- (H_λ^+) represent the boundary values of quasiperiodic solutions to the Dirac equation in the right (left) half-strip $x > x_0$ ($x < x_0$). To do this, rewrite the Dirac equation in the form

$$\partial_x \psi = \begin{pmatrix} -i\partial_y & m \\ m & i\partial_y \end{pmatrix} \psi. \tag{4.2}$$

If we put the initial condition $\psi(x_0, y) = g(x_0, y)$ with $g(x_0, y) \in H_\lambda^-$ (i.e. all $g_+(\theta_n) = 0$), the solution of (4.2) in the right half-strip is

$$\psi_{x>x_0} = \frac{2\pi}{\beta} \sum_{n\in\mathbb{Z}+\lambda} \frac{e^{imy \sinh\theta_n - m(x-x_0)\cosh\theta_n}}{2\cosh\theta_n} \begin{pmatrix} v_n^{1/2} & -v_n^{-1/2} \\ v_n^{-1/2} & v_n^{1/2} \end{pmatrix} \begin{pmatrix} 0 \\ g_-(\theta_n) \end{pmatrix}.$$

The convergence of this series is guaranteed by its convergence for $x = x_0$. The solution of (4.2) in the left half-strip can be constructed from the element of H_λ^+ in a similar fashion.

The Green function $G(z - z'; \lambda)$ on the cylinder without branchpoints provides a useful formula for the projections $Q_\pm$.

Proposition 4.1. *Consider the map $Q : H_\lambda^{1/2}(L_{x_0}) \to H_\lambda^1(\mathcal{C}\backslash L_{x_0})$, defined by*

$$(Qg)(z) = i \int_{L_{x_0}} G(z - z'; \lambda)\, \sigma_z\, g(y')dy', \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad g \in H_\lambda^{1/2}(L_{x_0}). \tag{4.3}$$

Then the boundary values on L_{x_0} of the restrictions of $(Qg)(z)$ to the left and right half-strip are equal to Q_+g and $-Q_-g$ respectively.

■ To prove the proposition, one has only to substitute in (4.3) the Fourier expansion (4.1) of g and the representations (3.2) and (3.4) of the Green function. □

Suppose no two branchpoints have the same first coordinate. Then one can isolate the branchcuts $b_1, \ldots, b_n$ in the open strips $S_1, \ldots, S_n$ (Fig. 5). The union $\bigcup_{j=1}^{n} S_j$ will be denoted by S. Consider the subspace of $H^1(\mathcal{C}\backslash\bar{S})$ consisting of functions ψ that satisfy on $\mathcal{C}\backslash\bar{S}$ the Dirac equation and appropriate quasiperiodicity conditions:

$$\psi(x, y+\beta) = \begin{cases} e^{2\pi i\lambda_0}\psi(x, y) \text{ for } x < x_1^L, \\ \exp\{2\pi i \sum_{j=0}^{k} \lambda_j\}\psi(x, y) \text{ for } x_k^R < x < x_{k+1}^L, \\ \exp\{2\pi i \sum_{j=0}^{n} \lambda_j\}\psi(x, y) \text{ for } x > x_n^R. \end{cases}$$

We will denote by W_{ext} the space of boundary values of such functions. It is a subspace of W, the space of all quasiperiodic $\mathbb{C}^2$-valued $H^{1/2}$ functions on ∂S:

$$W = H_{\lambda_0}^{1/2}(L_{x_1^L}) \oplus H_{\lambda_0+\lambda_1}^{1/2}(L_{x_1^R}) \oplus \cdots \oplus H_{\sum_{k=0}^n \lambda_k}^{1/2}(L_{x_n^R}). \tag{4.4}$$

Analogously, $W_{int} \subset W$ is defined as the space of boundary values of functions $g \in \mathcal{D}^{a,\lambda}$ that solve $D^{a,\lambda}g = 0$ on S.

The construction of the infinite-dimensional grassmannian in the next subsection heavily relies on the transversality of the subspaces W_{ext} and W_{int} in W. We postpone the proof of this fact; instead, let us explain how one can find the explicit formulae for the projections on these subspaces. Consider the restriction $g^{(i)} = g\big|_{\partial^L S_i \cup \partial^R S_i}$ of an element $g \in W$ to the boundary of the strip S_i. It is convenient to introduce the notation

$$g^{(i)} = \begin{pmatrix} g_{R,+}^{(i)} \\ g_{L,-}^{(i)} \end{pmatrix} \oplus \begin{pmatrix} g_{R,-}^{(i)} \\ g_{L,+}^{(i)} \end{pmatrix}.$$

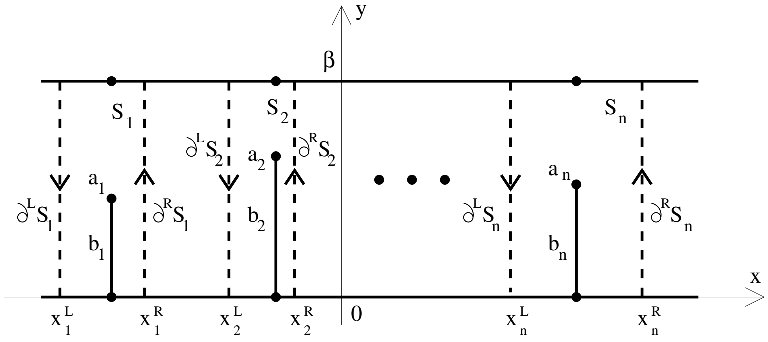

Fig. 5

Example. Assume for a moment that the strip S_i contains no branchcuts at all, i. e. both $g_L^{(i)}$ and $g_R^{(i)}$ obey the same quasiperiodicity conditions, say,

$$g_L^{(i)}(y+\beta) = e^{2\pi i\lambda} g_L^{(i)}(y), \qquad g_R^{(i)}(y+\beta) = e^{2\pi i\lambda} g_R^{(i)}(y).$$

Then the map

$$\begin{aligned}\tilde{Q}g^{(i)}(z) &= i \int\limits_{\partial^L S_i \cup \partial^R S_i} G(z-z';\lambda)\sigma_z\, g^{(i)}(y')dy' \\ &= \int\limits_{\partial^L S_i \cup \partial^R S_i} G_{\cdot,1}(z-z';\lambda)g_1^{(i)}(z')dz' + G_{\cdot,2}(z-z';\lambda)g_2^{(i)}(z')d\bar{z}' \qquad (4.5)\end{aligned}$$

defines a function that satisfies the Dirac equation in S_i. After a simple calculation involving Fourier representations of $g^{(i)}$ and of the Green function, one obtains the explicit formula

$$\begin{aligned}\tilde{Q}g^{(i)}(z) = \frac{2\pi}{\beta}\sum_{n\in\mathbb{Z}+\lambda} \frac{e^{m(x-x_i^R)\cosh\theta_n + imy\sinh\theta_n}}{2\cosh\theta_n} \begin{pmatrix} v_n^{1/2} & -v_n^{-1/2} \\ v_n^{-1/2} & v_n^{1/2}\end{pmatrix}\begin{pmatrix} g_{R,+}^{(i)}(\theta_n) \\ 0\end{pmatrix} \\ + \frac{2\pi}{\beta}\sum_{n\in\mathbb{Z}+\lambda} \frac{e^{-m(x-x_i^L)\cosh\theta_n + imy\sinh\theta_n}}{2\cosh\theta_n} \begin{pmatrix} v_n^{1/2} & -v_n^{-1/2} \\ v_n^{-1/2} & v_n^{1/2}\end{pmatrix}\begin{pmatrix} 0 \\ g_{L,-}^{(i)}(\theta_n)\end{pmatrix}.\end{aligned}$$

Passing to boundary values, we see that $\tilde{Q}$ induces a map on W. It is given by

$$\tilde{Q}: \begin{pmatrix} g_{R,+}^{(i)} \\ g_{L,-}^{(i)}\end{pmatrix} \oplus \begin{pmatrix} g_{R,-}^{(i)} \\ g_{L,+}^{(i)}\end{pmatrix} \mapsto \begin{pmatrix} g_{R,+}^{(i)} \\ g_{L,-}^{(i)}\end{pmatrix} \oplus \begin{pmatrix} 0 & \hat{\omega} \\ \hat{\omega} & 0\end{pmatrix}\begin{pmatrix} g_{R,+}^{(i)} \\ g_{L,-}^{(i)}\end{pmatrix},$$

where $(\hat{\omega}g)(\theta_n) = e^{-m(x_i^R - x_i^L)\cosh\theta_n} g(\theta_n)$ in Fourier representation. Furthermore, $\tilde{Q}$ is a projection onto the space of solutions to the Dirac equation on S_i. If $g_L^{(i)}$ and $g_R^{(i)}$ represent boundary values of a function f that belongs to this space, the one-form in (4.5) is closed, so the contour of integration can be shrunk to a small circle around z. Using the asymptotics of the Green function at $z' \to z$, one obtains $\tilde{Q}f(z) = f(z)$.

The generalization of the example we have just considered to the strip containing a branchcut leads to the main technical result of this subsection:

Theorem 4.2. *Suppose that* $G^{a,\lambda}(z,z')$ *is the one-point Green function*[2] *found in the previous section. Suppose that* $a \in S'$, $S' = \{(x,y) \in \mathcal{C} : x_L < x < x_R\}$. *Consider the function* g *on* $\partial S'$, *which satisfies* $g\big|_{\partial^L S'} \in H^{1/2}_{\lambda_0}(\partial^L S')$, $g\big|_{\partial^R S'} \in H^{1/2}_{\tilde{\lambda}}(\partial^R S')$. *Then the map*

$$P_{S'}(a)g(z) = \int\limits_{\partial^L S' \cup \partial^R S'} G^{a,\lambda}_{\cdot,1}(z,z')g_1(z')dz' + G^{a,\lambda}_{\cdot,2}(z,z')g_2(z')d\bar{z}' \tag{4.6}$$

defines a projection onto the space of functions $f \in \mathcal{D}^{a,\lambda}$ *that solve* $D^{a,\lambda} f = 0$ *on* S'. *The induced map of boundary values is determined by the following formula:*

$$P_{S'}(a) : \begin{pmatrix} g_{R,+} \\ g_{L,-} \end{pmatrix} \oplus \begin{pmatrix} g_{R,-} \\ g_{L,+} \end{pmatrix} \mapsto \begin{pmatrix} g_{R,+} \\ g_{L,-} \end{pmatrix} \oplus \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & \hat{\delta} \end{pmatrix} \begin{pmatrix} g_{R,+} \\ g_{L,-} \end{pmatrix}, \tag{4.7}$$

where

$$(\hat{\alpha} g)(\theta_l) = \frac{2\sin\pi\lambda_1}{\beta} \sum_{n\in\mathbb{Z}+\tilde{\lambda}} \frac{(v_l v_n)^{\lambda_1+\frac{1}{2}}}{1+v_l\, v_n} \frac{e^{-m(x_R-a_x)(\cosh\theta_l+\cosh\theta_n)-ima_y(\sinh\theta_l-\sinh\theta_n)}}{\cosh\theta_n}$$
$$\times e^{-\frac{i}{2}(\nu_l+\nu_n)-\frac{1}{2}(\eta_l+\eta_n)} g(\theta_n), \qquad l \in \mathbb{Z}+\tilde{\lambda}, \tag{4.8}$$
$$(\hat{\beta} g)(\theta_l) = \frac{2e^{-\pi i\lambda_1}\sin\pi\lambda_1}{\beta} \sum_{n\in\mathbb{Z}+\lambda_0} \frac{e^{-m(x_R-a_x)\cosh\theta_l+m(x_L-a_x)\cosh\theta_n-ima_y(\sinh\theta_l-\sinh\theta_n)}}{\cosh\theta_n}$$
$$\times \frac{v_l^{\lambda_1+\frac{1}{2}} v_n^{-\lambda_1+\frac{1}{2}}}{v_l - v_n} e^{-\frac{i}{2}(\nu_l-\nu_n)-\frac{1}{2}(\eta_l-\eta_n)} g(\theta_n), \qquad l \in \mathbb{Z}+\tilde{\lambda}, \tag{4.9}$$
$$(\hat{\gamma} g)(\theta_l) = -\frac{2e^{\pi i\lambda_1}\sin\pi\lambda_1}{\beta} \sum_{n\in\mathbb{Z}+\tilde{\lambda}} \frac{e^{m(x_L-a_x)\cosh\theta_l-m(x_R-a_x)\cosh\theta_n-ima_y(\sinh\theta_l-\sinh\theta_n)}}{\cosh\theta_n}$$
$$\times \frac{v_l^{-\lambda_1+\frac{1}{2}} v_n^{\lambda_1+\frac{1}{2}}}{v_l - v_n} e^{\frac{i}{2}(\nu_l-\nu_n)+\frac{1}{2}(\eta_l-\eta_n)} g(\theta_n), \qquad l \in \mathbb{Z}+\lambda_0, \tag{4.10}$$
$$(\hat{\delta} g)(\theta_l) = \frac{2\sin\pi\lambda_1}{\beta} \sum_{n\in\mathbb{Z}+\lambda_0} \frac{(v_l v_n)^{-\lambda_1+\frac{1}{2}}}{1+v_l\, v_n} \frac{e^{m(x_L-a_x)(\cosh\theta_l+\cosh\theta_n)-ima_y(\sinh\theta_l-\sinh\theta_n)}}{\cosh\theta_n}$$
$$\times e^{\frac{i}{2}(\nu_l+\nu_n)+\frac{1}{2}(\eta_l+\eta_n)} g(\theta_n), \qquad l \in \mathbb{Z}+\lambda_0, \tag{4.11}$$

and $\nu_l = \nu(\theta_l;\lambda)$, $\eta_l = \eta(\theta_l;\lambda)$.

■ The derivation of (4.2)–(4.11) can be carried out analogously to the previous example, using two more (in addition to (3.28–3.30)) representations of the one-point Green function:

---

[2] Here a denotes a single point, and not the collection $(a_1, \ldots, a_n)$. I hope this abuse of notation will not confuse the reader.

$$G^{a,\lambda}(z,z') = i\sin\pi\lambda_1 \sum_{l\in\mathbb{Z}+\lambda_0}\sum_{n\in\mathbb{Z}+\tilde{\lambda}} \frac{G(\theta_l;\lambda)H(\theta_n;1-\lambda)}{m\beta^2\cosh\theta_l\cosh\theta_n} \frac{e^{\theta_n+m(x-a_x)\cosh\theta_l+im(y-a_y)\sinh\theta_l}}{e^{\theta_l}-e^{\theta_n}}$$
$$\times e^{-m(x'-a_x)\cosh\theta_n-im(y'-a_y)\sinh\theta_n}\begin{pmatrix}-e^{\theta_l+\theta_n} & e^{\theta_l}\\ -e^{\theta_n} & 1\end{pmatrix} \quad \text{for } x<a_x<x', \tag{4.12}$$
$$G^{a,\lambda}(z,z') = i\sin\pi\lambda_1 \sum_{l\in\mathbb{Z}+\lambda_0}\sum_{n\in\mathbb{Z}+\tilde{\lambda}} \frac{G(-\theta_l;\lambda)H(-\theta_n;1-\lambda)}{m\beta^2\cosh\theta_l\cosh\theta_n} \frac{e^{-\theta_n-m(x-a_x)\cosh\theta_l+im(y-a_y)\sinh\theta_l}}{e^{-\theta_l}-e^{-\theta_n}}$$
$$\times e^{m(x'-a_x)\cosh\theta_n-im(y'-a_y)\sinh\theta_n}\begin{pmatrix}-e^{-\theta_l-\theta_n} & -e^{-\theta_l}\\ e^{-\theta_n} & 1\end{pmatrix} \quad \text{for } x>a_x>x'. \tag{4.13}$$

When one applies the Stokes theorem to prove the projection property, the contour of integration in (4.6) can be deformed into two small circles, around z and a. Using the expansions (2.7), (3.11) of the multivalued local solution to Dirac equation and Green function, one easily shows that the second integral vanishes.

Remark. Let us choose in $H_\lambda^{1/2}(L)$ a complete orthonormal family $\{\varphi_k\}$, say,

$$\varphi_k = \frac{1}{\sqrt{\beta}}e^{imy\sinh\theta_k}, \quad k\in\mathbb{Z}+\lambda.$$

With these functions, we can find Schmidt norms of $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\delta}$ and show that they are finite. For example,

$$\|\hat{\beta}\|_2^2 = \sum_{n,n'\in\mathbb{Z}} |\langle\hat{\beta}\varphi_{n+\lambda_0},\varphi_{n'+\tilde{\lambda}}\rangle|^2 = \sum_{l\in\mathbb{Z}+\tilde{\lambda}}\sum_{n\in\mathbb{Z}+\lambda_0}|\hat{\beta}(\theta_l,\theta_n)|^2,$$

where $\hat{\beta}(\theta_l,\theta_n)$ denotes the "kernel" of $\hat{\beta}$. This sum rapidly converges due to the exponential factors $e^{-m(x_R-a_x)\cosh\theta_l}$ and $e^{m(x_L-a_x)\cosh\theta_n}$ in $\hat{\beta}(\theta_l,\theta_n)$. Note, however, that in the limit $\beta\to\infty$, when $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\delta}$ become integral operators, $\hat{\beta}$ and $\hat{\gamma}$ no longer belong to the Schmidt class due to the singularities in the kernels.

We briefly outline the proof of the transversality of the subspaces W_{int} and W_{ext} in W, closely following Palmer's work [10]. Suppose that $f\in W$ is decomposed as $f=g+h$, with $g\in W_{int}$ and $h\in W_{ext}$. Theorem 2.1 guarantees the uniqueness of this decomposition, since the elements of $W_{int}\cap W_{ext}$ represent the boundary values of the functions from $\widetilde{\mathbf{W}}^{a,\lambda}$. It remains to prove only the existence.

In order for g to be in W_{int}, one should satisfy the conditions

$$\begin{pmatrix} g_{R,-}^{(i)} \\ g_{L,+}^{(i)}\end{pmatrix} = \begin{pmatrix}\hat{\alpha}_i & \hat{\beta}_i\\ \hat{\gamma}_i & \hat{\delta}_i\end{pmatrix}\begin{pmatrix} g_{R,+}^{(i)} \\ g_{L,-}^{(i)}\end{pmatrix}, \tag{4.14}$$

where $\hat{\alpha}_i$, $\hat{\beta}_i$, $\hat{\gamma}_i$, $\hat{\delta}_i$ are obtained from $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\delta}$ by the substitution

$$x_L\to x_i^L, \quad x_R\to x_i^R, \quad a\to a_i, \quad \lambda_0\to\sum_{k=0}^{i-1}\lambda_k, \quad \tilde{\lambda}\to\sum_{k=0}^{i}\lambda_k.$$

Another set of relations follows from the assumption that $h\in W_{ext}$. Indeed, $h_L^{(1)}$ should represent the boundary value of a solution to Dirac equation in the left half-strip $x<x_1^L$;

analogously, $h_R^{(n)}$ is the boundary value of a solution in the half-strip $x > x_n^R$. This leads to two relations,

$$h_{L,-}^{(1)} = 0, \qquad h_{R,+}^{(n)} = 0. \tag{4.15}$$

Next, the example we have considered shows that the functions $h_R^{(i)}$ and $h_L^{(i+1)}$ are boundary values of a solution in $x_i^R < x < x_{i+1}^L$, if

$$h_{L,-}^{(i+1)} = \hat{\omega}_i h_{R,-}^{(i)}, \qquad h_{R,+}^{(i)} = \hat{\omega}_i h_{L,+}^{(i+1)}, \qquad i = 1, \dots, n-1, \tag{4.16}$$

where $\hat{\omega}_i$ is obtained from $\hat{\omega}$ by the substitution

$$x_i^L \to x_i^R, \qquad x_i^R \to x_{i+1}^L, \qquad \lambda \to \sum_{k=0}^{i} \lambda_k.$$

One can transform (4.15) and (4.16) into the conditions on g. Using (4.14) to eliminate all $g_{R,-}^{(i)}$ and $g_{L,+}^{(i)}$, we obtain a system of equations,

$$g_{R,+}^{(i)} - \hat{\omega}_i \left(\hat{\gamma}_{i+1} g_{R,+}^{(i+1)} + \hat{\delta}_{i+1} g_{L,-}^{(i+1)} \right) = f_{R,+}^{(i)} - \hat{\omega}_i f_{L,+}^{(i+1)}, \quad i = 1, \dots, n-1, \tag{4.17}$$

$$g_{L,-}^{(i+1)} - \hat{\omega}_i \left(\hat{\alpha}_i g_{R,+}^{(i)} + \hat{\beta}_i g_{L,-}^{(i)} \right) = f_{L,-}^{(i+1)} - \hat{\omega}_i f_{R,-}^{(i)}, \qquad i = 1, \dots, n-1, \tag{4.18}$$

$$g_{L,-}^{(1)} = f_{L,-}^{(1)}, \qquad g_{R,+}^{(n)} = f_{R,+}^{(n)}. \tag{4.19}$$

If we introduce the notation

$$U_i = -\begin{pmatrix} \hat{\omega}_i \hat{\gamma}_{i+1} & \hat{\omega}_i \hat{\delta}_{i+1} \\ 0 & 0 \end{pmatrix}, \qquad V_i = -\begin{pmatrix} 0 & 0 \\ \hat{\omega}_i \hat{\alpha}_i & \hat{\omega}_i \hat{\beta}_i \end{pmatrix}, \qquad i = 1, \dots, n-1,$$

$$\tilde{g}_j = \begin{pmatrix} g_{R,+}^{(j)} \\ g_{L,-}^{(j)} \end{pmatrix}, \qquad F_k = \begin{pmatrix} f_{R,+}^{(k)} - \hat{\omega}_k f_{L,+}^{(k+1)} \\ f_{L,-}^{(k)} - \hat{\omega}_{k-1} f_{R,-}^{(k-1)} \end{pmatrix}, \qquad j = 1, \dots, n, \quad k = 2, \dots, n-1,$$

$$F_1 = \begin{pmatrix} f_{R,+}^{(1)} - \hat{\omega}_1 f_{L,+}^{(2)} \\ f_{L,-}^{(1)} \end{pmatrix}, \qquad F_n = \begin{pmatrix} f_{R,+}^{(n)} \\ f_{L,-}^{(n)} - \hat{\omega}_{n-1} f_{R,-}^{(n-1)} \end{pmatrix},$$

the system (4.17)–(4.19) can be rewritten as

$$\begin{pmatrix} \mathbf{1} & U_1 & 0 & . & 0 \\ V_1 & \mathbf{1} & U_2 & . & 0 \\ 0 & V_2 & \mathbf{1} & . & . \\ . & . & . & . & U_{n-1} \\ 0 & 0 & . & V_{n-1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \tilde{g}_1 \\ . \\ . \\ . \\ \tilde{g}_n \end{pmatrix} = \begin{pmatrix} F_1 \\ . \\ . \\ . \\ F_n \end{pmatrix}. \tag{4.20}$$

The operator, standing in the LHS of (4.20), represents a compact perturbation of the identity, and thus is Fredholm of index zero. Since to every nontrivial element of its kernel there corresponds a nontrivial element of $W_{int} \cap W_{ext}$, the kernel should be zero. Therefore, this operator is invertible and can be used to construct the decomposition $f = g + h$ explicitly.

4.2. Grassmannian, det**-bundle and its trivialization.* First we introduce several important definitions, following Segal and Wilson [18] (further details can be found in [14, 19]).

Definition 4.3. *Suppose we have a complex Hilbert space H with a given decomposition $H = H_+ \oplus H_-$. The Grassmannian $Gr(H)$ is a set of all closed subspaces $V \subset H$ such that*

- *the projection* $\mathrm{pr}_+ : V \to H_+$ *along* H_- *is a Fredholm operator;*
- *the projection* $\mathrm{pr}_- : V \to H_-$ *along* H_+ *is a Hilbert-Schmidt operator.*

The first requirement means that the codimension of $V \cap H_+$ is finite in both V and H_+. The connected components of the Grassmannian are distinguished by the value of the index of pr_+. We shall work only with the component $Gr_0(H)$ that corresponds to zero index.

Definition 4.4. *The invertible linear map $v : H_+ \to V$ is called an admissible frame for the subspace $V \in Gr_0(H)$ if $\mathrm{pr}_+ \circ v : H_+ \to H_+$ is a trace class perturbation of the identity. The fiber of the* det**-bundle over V consists of the equivalence classes of pairs (v, α), where v is an admissible frame, α is a complex number and $(v_1, \alpha_1) \sim (v_2, \alpha_2)$ if $\alpha_1 = \alpha_2 \det\left(v_2^{-1} v_1\right)$. The canonical section of the* det**-bundle is given by $\sigma : V \mapsto \left(v, \det(\mathrm{pr}_+ \circ v)\right)$.*

In the work of Segal and Wilson, as well as in almost all subsequent papers on the subject, the Hilbert space H is the space of all square-integrable complex-valued functions on the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. H_+ and H_- are spanned by the elements $\{z^k\}$ with $k \geq 0$ and $k < 0$, respectively.

We are interested in a more complicated model of the Grassmannian. H is identified with the space W (see (4.4)) of square-integrable quasiperiodic $\mathbb{C}^2$-valued functions on the boundary ∂S. Let us fix a collection of points $a^0 = (a_1^0, \ldots, a_n^0)$ such that $a_j^0 \in S_j$, $j = 1, \ldots, n$. Then one can define the Grassmannian $Gr(W)$ with respect to the splitting $W = W_{int}(a^0) \oplus W_{ext}$. The crucial observation, similar to one made by Palmer in [10], is that $W_{int}(a) \in Gr_0(W)$.

Now introduce two admissible frames for the subspace $W_{int}(a)$. The first one, which will be denoted as $P(a) : W_{int}(a^0) \to W_{int}(a)$, represents the projection of $W_{int}(a^0)$ on $W_{int}(a)$ along W_{ext}. It is easy to guess that $P(a)$ inverts pr_+. Thus the canonical section can be written as $\sigma : W_{int}(a) \mapsto (P(a), 1)$. The second admissible frame, $F(a) : W_{int}(a^0) \to W_{int}(a)$, is the restriction to $W_{int}(a^0)$ of the direct sum of the appropriate one-point projections (4.6)–(4.7): $F(a) = \left(P_{S_1}(a_1) \oplus \cdots \oplus P_{S_n}(a_n)\right)\Big|_{W_{int}(a^0)}$. It defines a second, trivializing section $\vartheta : W_{int}(a) \mapsto (F(a), 1)$. The determinant of the Dirac operator $D^{a,\lambda}$, or the τ-function, is determined from the comparison of the two sections,

$$\tau(a, a^0) = \frac{\sigma(W_{int}(a))}{\vartheta(W_{int}(a))} = \det\left(P(a)^{-1} F(a)\right). \tag{4.21}$$

Remark. This ideology originates from the work [9], where the isomonodromic τ-function, associated to a fuchsian system on $\mathbb{CP}^1$, was interpreted as the determinant of a Cauchy-Riemann operator, whose domain incorporates functions with specified branching.

In order to calculate $\tau(a, a^0)$ more explicitly, let us use several results and notations from the previous subsection. Suppose that $f \in W_{int}(a^0)$, $g \in W_{int}(a)$, then

$$f^{(j)} = \tilde{f}_j \oplus N_j\left(a^0\right)\tilde{f}_j, \quad g^{(j)} = \tilde{g}_j \oplus N_j(a)\,\tilde{g}_j, \qquad N_j(a) = \begin{pmatrix} \hat{\alpha}_j(a) & \hat{\beta}_j(a) \\ \hat{\gamma}_j(a) & \hat{\delta}_j(a) \end{pmatrix}. \tag{4.22}$$

The functions f and g can be represented by columns

$$f = (\tilde{f}_1 \ldots \tilde{f}_n)^T, \qquad g = (\tilde{g}_1 \ldots \tilde{g}_n)^T.$$

The map $F(a) : W_{int}(a^0) \to W_{int}(a)$ is given in these coordinates by the identity transformation. To obtain the representation of $P(a)^{-1}$, for each $g \in W_{int}(a)$ one should find a function $f \in W_{int}(a^0)$ such that $g = f - h$ with $h \in W_{ext}$. This amounts to the same calculation as we have done earlier in (4.15)–(4.20). Taking into account the additional condition (4.22) on f, one finally obtains

$$(\mathbf{1} + M(a))\,g = \left(\mathbf{1} + M(a^0)\right) f,$$

where

$$M(a) = \begin{pmatrix} 0 & U_1(a) & 0 & . & 0 \\ V_1(a) & 0 & U_2(a) & . & 0 \\ 0 & V_2(a) & 0 & . & . \\ . & . & . & . & U_{n-1}(a) \\ 0 & 0 & . & V_{n-1}(a) & 0 \end{pmatrix}.$$

Therefore, the τ-function is equal to

$$\tau(a, a^0) = \det\left\{(\mathbf{1} + M(a))\left(\mathbf{1} + M(a^0)\right)^{-1}\right\}. \tag{4.23}$$

In fact, one can derive an even more convenient representation. Let us introduce the matrix

$$\tilde{M}(a) = \begin{pmatrix} 0 & \tilde{U}_1(a) & 0 & . & 0 \\ \tilde{V}_1(a) & 0 & \tilde{U}_2(a) & . & 0 \\ 0 & \tilde{V}_2(a) & 0 & . & . \\ . & . & . & . & \tilde{U}_{n-1}(a) \\ 0 & 0 & . & \tilde{V}_{n-1}(a) & 0 \end{pmatrix}$$

with

$$\tilde{U}_j = \begin{pmatrix} -\hat{\omega}_j\hat{\gamma}_{j+1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{V}_j = \begin{pmatrix} 0 & 0 \\ 0 & -\hat{\omega}_j\hat{\beta}_j \end{pmatrix}, \qquad j = 1, \ldots, n-1.$$

The matrix $\mathbf{1} + \tilde{M}(a)$ is a product of an upper triangular and a lower triangular matrix with identities on the diagonals. Thus we have

$$\det\left\{\left(\mathbf{1} + \tilde{M}(a^0)\right)\left(\mathbf{1} + \tilde{M}(a)\right)^{-1}\right\} = 1.$$

Multiplying the RHS of (4.23) by this determinant, one finds

$$\tau(a,a^0)=\det\left\{\left(\mathbf{1}+\tilde{M}(a)\right)^{-1}(\mathbf{1}+M(a))\left(\mathbf{1}+M(a^0)\right)^{-1}\left(\mathbf{1}+\tilde{M}(a^0)\right)\right\}=\frac{\tau(a)}{\tau(a^0)},$$

where

$$\tau(a)=\det\left\{\left(\mathbf{1}+\tilde{M}(a)\right)^{-1}(\mathbf{1}+M(a))\right\}. \tag{4.24}$$

Example. Consider the simplest nontrivial situation, when there are only two branching points on the cylinder. Let us introduce the notation

$$\tilde{\lambda}=\lambda_0+\lambda_1,\quad \bar{\lambda}=\lambda_0+\lambda_1+\lambda_2,\qquad a_x=(a_2)_x-(a_1)_x,\quad a_y=(a_2)_y-(a_1)_y.$$

In the case $n=2$ the inverse matrix $\left(\mathbf{1}+\tilde{M}(a)\right)^{-1}$ looks particularly simple,

$$\left(\mathbf{1}+\tilde{M}(a)\right)^{-1}=\begin{pmatrix}\mathbf{1} & -\tilde{U}_1(a)\\ -\tilde{V}_1(a) & \mathbf{1}\end{pmatrix}.$$

Using this formula, one can find that the two-point τ-function is given by

$$\tau(a)=\det\left(\mathbf{1}-K\right), \tag{4.25}$$

where the operator $K=\hat{\omega}_1\hat{\alpha}_1\hat{\omega}_1\hat{\delta}_2$ can be represented by the infinite-dimensional matrix with the entries

$$\begin{aligned}K_{mn}=&\;\frac{4\sin\pi\lambda_1\sin\pi\lambda_2}{\beta^2}\,\frac{(v_m v_n)^{\frac{\lambda_1-\lambda_2+1}{2}}}{\sqrt{\cosh\theta_m\cosh\theta_n}}\\ &\times\sum_{l\in\mathbb{Z}+\tilde{\lambda}}\frac{v_l^{\lambda_1-\lambda_2+1}e^{-m|a_x|\frac{\cosh\theta_m+2\cosh\theta_l+\cosh\theta_n}{2}+ima_y\frac{\sinh\theta_m-2\sinh\theta_l+\sinh\theta_n}{2}+\frac{\rho(\theta_m)+2\rho(\theta_l)+\rho(\theta_n)}{2}}}{(1+v_m v_l)(1+v_l v_n)\cosh\theta_l}.\end{aligned} \tag{4.26}$$

The indices take on the values $m,n\in\mathbb{Z}+\tilde{\lambda}$ and

$$2\rho(\theta)=\eta(\theta;\tilde{\lambda},\bar{\lambda})-\eta(\theta;\lambda_0,\tilde{\lambda})+i\nu(\theta;\tilde{\lambda},\bar{\lambda})-i\nu(\theta;\lambda_0,\tilde{\lambda}).$$

One can also write K as

$$\begin{aligned}K&=4\sin\pi\lambda_1\sin\pi\lambda_2\cdot VV^T,\\ V_{mn}&=\frac{1}{\beta}\,\frac{(v_m v_n)^{\frac{\lambda_1-\lambda_2+1}{2}}e^{-m|a_x|\frac{\cosh\theta_m+\cosh\theta_n}{2}+ima_y\frac{\sinh\theta_m-\sinh\theta_n}{2}+\frac{\rho(\theta_m)+\rho(\theta_n)}{2}}}{\sqrt{\cosh\theta_m\cosh\theta_n}(1+v_m v_n)},\qquad m,n\in\mathbb{Z}+\tilde{\lambda}.\end{aligned}$$

These explicit formulae for the τ-function are in some sense a reward for the technical work put in the calculation of the element of canonical basis on the 1-punctured cylinder (Theorem 2.3). It would be interesting to compare them with the correlation functions of twisted fields, calculated in the lattice regularization of the Dirac theory on the plane [4].

Remark that the final answer (4.25)–(4.26) for the two-point τ-function is independent of the choice of localization (coordinates of edges of the strips $S_1,\ldots,S_n$). To show that this is true in the general case, we prove

Proposition 4.5. *The logarithmic derivatives of the τ-function (4.21) are given by*

$$d \ln \tau(a, a^0) = \frac{m}{2} \sum_{\nu=1}^{n} \left\{ a_{1/2}^{(\nu)}(\tilde{\mathbf{w}}_\nu(\lambda))\, da_\nu + \overline{a_{1/2}^{(\nu)}(\tilde{\mathbf{w}}_\nu(-\lambda))}\, d\bar{a}_\nu \right\}. \tag{4.27}$$

■ Consider the n-point Green function $G^{a,\lambda}(z, z')$ and construct the map $\tilde{P}(a) : W \to W_{int}(a)$ in the following way:

$$\tilde{P}(a) f(z) = \int_{\partial S} G^{a,\lambda}_{\cdot,1}(z, z') f_1(z') dz' + G^{a,\lambda}_{\cdot,2}(z, z') f_2(z') d\bar{z}'. \tag{4.28}$$

It is easy to see that this map defines the projection on $W_{int}(a)$ along W_{ext}. Indeed, let us write the function $f(z)$ as $f = g + h$, with $g \in W_{int}(a)$ and $h \in W_{ext}$. The form in the integral $\tilde{P}(a)g(z)$ is closed, thus each piece ∂S_μ of the integration contour can be shrunk up to two small circles, around $z' = z$ and $z' = a_\mu$. Computing the residues, we obtain $\tilde{P}(a)g(z) = g(z)$. In a similar fashion one also shows that $\tilde{P}(a)h(z) = 0$.

It is clear that the admissible frame $P(a) : W_{int}(a^0) \to W_{int}(a)$ and the projection $\mathrm{pr}_+ : W_{int}(a) \to W_{int}(a^0)$ are the restrictions

$$P(a) = \tilde{P}(a)\Big|_{W_{int}(a^0)}, \qquad \mathrm{pr}_+ = \tilde{P}(a^0)\Big|_{W_{int}(a)}.$$

Let us analogously consider the map $\tilde{F}(a) : W \to W_{int}(a)$, which is by definition the direct sum of the one-point projections, $\tilde{F}(a) = P_{S_1}(a_1) \oplus \cdots \oplus P_{S_n}(a_n)$. The second admissible frame that we have used, $F(a) : W_{int}(a^0) \to W_{int}(a)$, is the restriction $F(a) = \tilde{F}(a)\Big|_{W_{int}(a^0)}$. Its inverse, which we denote as $F(a^0) : W_{int}(a) \to W_{int}(a^0)$, is given by $F(a^0) = \tilde{F}(a^0)\Big|_{W_{int}(a)}$.

Therefore, differentiating (4.21), one obtains

$$\begin{aligned} d \ln \tau(a, a^0) &= -\mathrm{Tr} \left\{ d\left(F(a)^{-1} P(a) \right) P(a)^{-1} F(a) \right\} \\ &= -\mathrm{Tr} \left\{ F(a^0) d(P(a)) \mathrm{pr}_+ F(a) \right\}. \end{aligned}$$

Recall that the traces in the last formula, and the determinant in (4.21) as well, are calculated on the subspace $W_{int}(a^0)$. However, since the range of both $F(a^0)$ and $\tilde{F}(a^0)$ is $W_{int}(a^0)$, we can forget about this restriction and replace under the last trace $P(a)$ by $\tilde{P}(a)$, $F(a)$ by $\tilde{F}(a)$, pr_+ by $\tilde{P}(a^0)$ and $F(a^0)$ by $\tilde{F}(a^0)$. If we also use the relations

$$\tilde{P}(a)(\mathbf{1} - \tilde{P}(a^0)) = 0, \qquad \tilde{F}(a)(\mathbf{1} - \tilde{F}(a^0)) = 0,$$

the trace becomes

$$d \ln \tau(a, a^0) = -\mathrm{Tr}_W \Big\{ \tilde{F}(a^0) d(\tilde{P}(a)) \tilde{P}(a^0) \tilde{F}(a) \Big\} = -\mathrm{Tr}_W \Big\{ d(\tilde{P}(a)) \tilde{F}(a) \Big\}.$$

Taking into account the explicit form (4.28) of $\tilde{P}(a)$ and the formulae (3.23)–(3.24) for the derivatives of the Green function, one can show that $d\tilde{P}(a)$ is an integral operator with degenerate kernel. Then we have, for example,

$$\partial_{a_\nu} \ln \tau(a, a^0) = \frac{im^2}{8 \sin \pi \lambda_\nu} \sum_{\mu=1}^{n} \int_{\partial S_\mu} \left\{ \Big(\tilde{\mathbf{w}}_\nu(z, -\lambda) \Big)_1 \Big(P_{S_\mu}(a_\mu) \tilde{\mathbf{w}}_\nu(z, \lambda) \Big)_1 dz \right.$$
$$\left. + \Big(\tilde{\mathbf{w}}_\nu(z, -\lambda) \Big)_2 \Big(P_{S_\mu}(a_\mu) \tilde{\mathbf{w}}_\nu(z, \lambda) \Big)_2 d\bar{z} \right\}.$$

Again applying the Stokes theorem, each contour ∂S_μ can be shrunk to a small circle around a_μ. Only one circle, around a_ν, gives a non-zero contribution, which can be calculated using the asymptotics of the one-point Green function. At the end of this calculation one finds

$$\partial_{a_\nu} \ln \tau(a, a^0) = \frac{m}{2}\, a_{1/2}^{(\nu)}(\tilde{\mathbf{w}}_\nu(\lambda)), \qquad \partial_{\bar{a}_\nu} \ln \tau(a, a^0) = \frac{m}{2}\, \overline{a_{1/2}^{(\nu)}(\tilde{\mathbf{w}}_\nu(-\lambda))},$$

as claimed. □

5. Deformation Equations

Let us now find the differential equations satisfied by the elements (3.20), using the idea that we have already exploited in the calculation of the derivative of the Green function. For example, consider the solution $\tilde{\mathbf{w}}_\mu(\lambda)$ and differentiate it with respect to a_ρ. We obtain again a solution of the Dirac equation but with more singular local expansions at the branchpoints:

$$\partial_{a_\rho} \tilde{\mathbf{w}}_\mu(\lambda)[a_\nu] = \sum_{k>0} \left\{ \partial_{a_\rho} a_k^{(\nu)}(\tilde{\mathbf{w}}_\mu(\lambda))\; w_{k+\lambda_\nu}[a_\nu] + \partial_{a_\rho} b_k^{(\nu)}(\tilde{\mathbf{w}}_\mu(\lambda))\; w^*_{k-\lambda_\nu}[a_\nu] \right\}$$
$$- \frac{m}{2}\, \delta_{\rho\nu} \left[\delta_{\mu\nu} w_{-3/2+\lambda_\nu}[a_\nu] + \sum_{k>0} \left\{ a_k^{(\nu)}(\tilde{\mathbf{w}}_\mu(\lambda))\; w_{k-1+\lambda_\nu}[a_\nu] \right.\right.$$
$$\left.\left. + b_k^{(\nu)}(\tilde{\mathbf{w}}_\mu(\lambda))\; w^*_{k+1-\lambda_\nu}[a_\nu] \right\} \right].$$

Adding the appropriate linear combination of $\{\tilde{\mathbf{w}}_\eta(\lambda)\}$, $\{\partial_z \tilde{\mathbf{w}}_\eta(\lambda)\}$ and $\{\partial_{\bar{z}} \tilde{\mathbf{w}}_\eta(\lambda)\}$ ($\eta = 1, \ldots, n$), one can annihilate the coefficients near the "extra" terms $w_{-3/2+\lambda_\nu}$ and $w_{-1/2+\lambda_\nu}$. Then the result will vanish identically, since it is clearly in $\widetilde{\mathbf{W}}^{a,\lambda}$. This observation can be written in the following general form:

$$d_{a,\bar{a}} \vec{\mathbf{w}}(\lambda) = (\Phi\, \partial_z + \Phi^* \partial_{\bar{z}} + \Psi)\, \vec{\mathbf{w}}(\lambda). \tag{5.1}$$

Here $d_{a,\bar{a}} = \sum_{j=1}^{n} (da_j \cdot \partial_{a_j} + d\bar{a}_j \cdot \partial_{\bar{a}_j})$ denotes the differential with respect to the positions of the singularities, Φ, Φ^* and Ψ are matrix-valued one-forms, and $\vec{\mathbf{w}}(\lambda) = (\tilde{\mathbf{w}}_1(\lambda) \ldots \tilde{\mathbf{w}}_n(\lambda))^T$.

Let us introduce the notation

$$C_j = \left[a^{(\nu)}_{j+1/2}(\tilde{\mathbf{w}}_\mu(\lambda))\right]_{\mu,\nu=1,\dots,n}, \quad C^*_j = \left[b^{(\nu)}_{j-1/2}(\tilde{\mathbf{w}}_\mu(\lambda))\right]_{\mu,\nu=1,\dots,n}, \qquad j \in \mathbb{Z}. \tag{5.2}$$

In particular, one has $C_0 = \mathbf{1}$ and $C_j = C^*_j = \mathbf{0}$ for $j < 0$. The system (5.1), being rewritten in terms of $\{C_j\}$, $\{C^*_j\}$, amounts to

$$dC_j - \frac{m}{2}\, C_{j+1} dA - \frac{m}{2}\, C_{j-1} d\bar{A} = \frac{m}{2}\, \Phi\, C_{j+1} + \frac{m}{2}\, \Phi^* C_{j-1} + \Psi C_j, \tag{5.3}$$

$$dC^*_j - \frac{m}{2}\, C^*_{j-1} dA - \frac{m}{2}\, C^*_{j+1} d\bar{A} = \frac{m}{2}\, \Phi\, C^*_{j-1} + \frac{m}{2}\, \Phi^* C^*_{j+1} + \Psi C^*_j, \tag{5.4}$$

where $dA = (\delta_{\mu\nu} da_\nu)_{\mu,\nu=1,\dots,n}$ and $d\bar{A} = (\delta_{\mu\nu} d\bar{a}_\nu)_{\mu,\nu=1,\dots,n}$.

Note that the expansion coefficients obey a set of algebraic relations. To derive them, let us first consider two multivalued solutions to Dirac equation, u and v, that are square integrable at $|x| \to \infty$ and have the local expansions (2.3) at the singularities. We shall assume that there exists a negative half-integer number k_0 such that

$$a^{(\nu)}_k(u) = b^{(\nu)}_k(u) = a^{(\nu)}_k(v) = b^{(\nu)}_k(v) = 0, \qquad \nu = 1, \dots, n,$$

for all $k < k_0$. Using (2.9) and Stokes theorem, calculate in two different ways the integral

$$\frac{m^2}{2} \int\limits_{\mathcal{C} \setminus \bigcup_\nu D_\varepsilon(a_\nu)} (\bar{u}_1 v_1 + \bar{u}_2 v_2)\, i dz \wedge d\bar{z} = im \oint\limits_{\bigcup_\nu \partial D_\varepsilon(a_\nu)} \bar{u}_2 v_1\, dz = -im \oint\limits_{\bigcup_\nu \partial D_\varepsilon(a_\nu)} \bar{u}_1 v_2\, d\bar{z}.$$

Comparing the asymptotics of the corresponding boundary integrals as $\varepsilon \to 0$, one obtains

$$\sum_{\nu=1}^{n} \sum_{k \in \mathbb{Z}+\frac{1}{2}} \left\{ \overline{b^{(\nu)}_k(u)}\, a^{(\nu)}_{-k}(v) - \overline{a^{(\nu)}_{-k}(u)}\, b^{(\nu)}_k(v) \right\} (-1)^{k-1/2} \sin \pi \lambda_\nu = 0. \tag{5.5}$$

If we now put $u = \tilde{\mathbf{w}}_\mu$, $v = \tilde{\mathbf{w}}_\rho$, then (5.5) leads to the relation (analogous to (2.13))

$$\overline{b^{(\rho)}_{1/2}(\tilde{\mathbf{w}}_\mu)}\, \sin \pi \lambda_\rho = b^{(\mu)}_{1/2}(\tilde{\mathbf{w}}_\rho)\, \sin \pi \lambda_\mu,$$

or, in matrix notation,

$$C^*_0 \, \sin \pi \Lambda = \left[\overline{C^*_0} \sin \pi \Lambda\right]^T, \qquad \Lambda = \left(\delta_{\mu\nu} \lambda_\nu\right)_{\mu,\nu=1,\dots,n}. \tag{5.6}$$

On the other hand, the substitution $u = \tilde{\mathbf{w}}_\mu(\lambda)$, $v = \tilde{\mathbf{w}}^*_\nu(-\lambda)$ gives

$$C_1(\lambda) \sin \pi \Lambda = [C_1(-\lambda) \sin \pi \Lambda]^T. \tag{5.7}$$

Finally, observe that the entries of the n-dimensional vector $\partial_{\bar{z}} \vec{\mathbf{w}}(\lambda) - \frac{m}{2}\, C^*_0(\lambda)\, \vec{\mathbf{w}}^*(-\lambda)$ belong to $\widetilde{\mathbf{W}}^{a,\lambda}$ and thus are all equal to zero. This gives two more relations,

$$C^*_0(\lambda) \overline{C_1(-\lambda)} = C^*_1(\lambda), \qquad C^*_0(\lambda) \overline{C^*_0(-\lambda)} = \mathbf{1}. \tag{5.8}$$

It is easy to see that for positive λ_μ, λ_ν both $\tilde{\mathbf{w}}_\mu(\lambda), \tilde{\mathbf{w}}_\nu(\lambda) \in \mathbf{W}^{a,\lambda}$, so we can calculate the inner product:

$$\langle \tilde{\mathbf{w}}_\mu(\lambda), \tilde{\mathbf{w}}_\nu(\lambda) \rangle = -4\,\overline{b^{(\nu)}_{1/2}(\tilde{\mathbf{w}}_\mu(\lambda))}\,\sin \pi\lambda_\nu.$$

This shows that the submatrix of $C_0^*(\lambda)\sin\pi\Lambda$, associated with the indices corresponding to positive $\{\lambda_\rho\}$, is negative definite. In a similar fashion, one finds

$$\langle \tilde{\mathbf{w}}^*_\mu(-\lambda), \tilde{\mathbf{w}}^*_\nu(-\lambda) \rangle = 4\overline{b^{(\mu)}_{1/2}(\tilde{\mathbf{w}}_\nu(-\lambda))}\,\sin \pi\lambda_\mu$$

for negative λ_μ, λ_ν. Combining this formula with the second relation in (5.8), one can prove that the submatrix of $\left(C_0^*(\lambda)\right)^{-1}\sin\pi\Lambda$, that corresponds to the "negative" indices, is positive definite.

Let us return to deformation equations (5.3) and (5.4). In order to determine the unknown matrix-valued forms Φ and Φ^*, let $j=-1$. One then finds

$$\Phi = -dA, \qquad \Phi^* = -C_0^* d\bar{A}\left(C_0^*\right)^{-1}. \tag{5.9}$$

Specializing to the case $j=0$, we calculate the form Ψ and obtain a matrix equation,

$$\Psi = \frac{m}{2}[dA, C_1] = dC_0^*\left(C_0^*\right)^{-1} + \frac{m}{2}\left[C_0^* d\bar{A}\left(C_0^*\right)^{-1}, C_1^*\left(C_0^*\right)^{-1}\right]. \tag{5.10}$$

For $j=1$, higher order coefficients arise. However, the "antiholomorphic" part of (5.3) and "holomorphic" part of (5.4) comprise only the coefficients that are already involved:

$$d_{\bar{a}}C_1 + \frac{m}{2}\left[C_0^* d\bar{A}, \left(C_0^*\right)^{-1}\right] = 0, \tag{5.11}$$

$$d_a C_1^* + \frac{m}{2}[dA, C_0^*] - \frac{m}{2}[dA, C_1]\,C_1^* = 0, \tag{5.12}$$

where $d_a = \sum_{j=1}^{n} da_j \cdot \partial_{a_j}$ and $d_{\bar{a}} = \sum_{j=1}^{n} d\bar{a}_j \cdot \partial_{\bar{a}_j}$ In addition, the diagonal part of (5.3) implies

$$d_a\,\mathrm{diag}\,C_1 = \frac{m}{2}\,\mathrm{diag}\,\left([dA, C_1]\,C_1\right). \tag{5.13}$$

In order to write the deformation equations in more compact and standard form, introduce the notation

$$G = C_0^*\sin\pi\Lambda, \qquad \Theta = \frac{m}{2}[dA, C_1], \qquad \Theta^\dagger = \bar{\Theta}^T. \tag{5.14}$$

Using the symmetry relations (5.6)–(5.8), one can show that (5.10) transforms into

$$dG = \Theta G + G\Theta^\dagger. \tag{5.15}$$

Instead of Eqs. (5.11) and (5.12) we have two conjugate relations

$$d_{\bar{a}}\,C_1 = \frac{m}{2}[d\bar{A}, G]\,G^{-1}, \qquad d_a\,\overline{C}_1 = \frac{m}{2}[dA, \overline{G}]\,\overline{G}^{-1}, \tag{5.16}$$

and the last Eq. (5.13) can be rewritten as

$$d_a \operatorname{diag} C_1 = \operatorname{diag}(\Theta\, C_1). \tag{5.17}$$

We easily find from (5.14) and (5.15) that $\det G = \text{const}$. It is also very instructive to deduce the closedness of the 1-form

$$\Omega = \frac{m}{2} \sum_{\nu=1}^{n} \left\{ a_{1/2}^{(\nu)}(\tilde{\mathbf{w}}_\nu(\lambda))\, da_\nu + \overline{a_{1/2}^{(\nu)}(\tilde{\mathbf{w}}_\nu(-\lambda))}\, d\bar{a}_\nu \right\} = \frac{m}{2} \operatorname{Tr}(C_1 dA + \overline{C}_1 d\bar{A}),$$

standing in the RHS of (4.27), from the deformation equations. Indeed,

$$\begin{aligned} d\Omega &= \frac{m}{2} \operatorname{Tr}\left(\Theta\, C_1 \wedge dA + \frac{m}{2}[d\bar{A}, G]\, G^{-1} \wedge dA + \overline{\Theta}\,\overline{C}_1 \wedge d\bar{A} + \frac{m}{2}[dA, \overline{G}]\,\overline{G}^{-1} \wedge d\bar{A}\right) \\ &= -\frac{m^2}{4} \operatorname{Tr}\left(C_1 dA \wedge C_1 dA + \overline{C}_1 d\bar{A} \wedge \overline{C}_1 d\bar{A} + G d\bar{A} \wedge G^{-1} dA + \overline{G} dA \wedge \overline{G}^{-1} d\bar{A}\right) = 0, \end{aligned}$$

so the form Ω does represent the differential of a function.

Example. As an illustration, let us find the explicit form of the deformation equations in the case $n = 2$. Suppose that $\lambda_1 > 0$ and $\lambda_2 < 0$. Then $G_{11} < 0$, $\det G < 0$, and the matrix G can be parametrized in the following way:

$$G = \chi \begin{pmatrix} -e^{\eta} \sin\psi & e^{i\varphi} \cos\psi \\ e^{-i\varphi} \cos\psi & e^{-\eta} \sin\psi \end{pmatrix}, \qquad \chi, \eta, \psi, \varphi \in \mathbb{R},$$

where $0 < \psi < \pi$, $\chi > 0$. We shall also denote

$$C_1 = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}, \qquad q = m(a_2 - a_1)/2, \quad \bar{q} = m(\bar{a}_2 - \bar{a}_1)/2.$$

From (5.15) one obtains

$$\begin{aligned} \frac{\partial G}{\partial q} &= -\chi^{-1} \begin{pmatrix} \Lambda_{12} e^{-i\varphi} \cos\psi & \Lambda_{12} e^{-\eta} \sin\psi \\ \Lambda_{21} e^{\eta} \sin\psi & -\Lambda_{21} e^{i\varphi} \cos\psi \end{pmatrix}, \\ \frac{\partial G}{\partial \bar{q}} &= -\chi^{-1} \begin{pmatrix} \overline{\Lambda}_{12} e^{i\varphi} \cos\psi & \overline{\Lambda}_{21} e^{\eta} \sin\psi \\ \overline{\Lambda}_{12} e^{-\eta} \sin\psi & -\overline{\Lambda}_{21} e^{-i\varphi} \cos\psi \end{pmatrix}. \end{aligned}$$

This leads to the relations

$$\begin{aligned} &\partial_q \varphi = i \operatorname{tg}^2\psi\, \partial_q \eta, \qquad \partial_{\bar{q}} \varphi = -i \operatorname{tg}^2\psi\, \partial_{\bar{q}} \eta, \\ &\Lambda_{12} = e^{\eta + i\varphi}\left(\partial_q \psi - i \operatorname{ctg}\psi\, \partial_q \varphi\right), \qquad \Lambda_{21} = e^{-\eta - i\varphi}\left(\partial_q \psi + i \operatorname{ctg}\psi\, \partial_q \varphi\right). \end{aligned}$$

Next, the first formula in (5.16) implies that

$$\frac{\partial C_1}{\partial \bar{q}} = -\begin{pmatrix} \cos^2\psi & e^{\eta + i\varphi} \cos\psi \sin\psi \\ e^{-\eta - i\varphi} \cos\psi \sin\psi & -\cos^2\psi \end{pmatrix}. \tag{5.18}$$

The off-diagonal part of this relation leads to a system of coupled differential equations,

$$\begin{cases} \partial_{q\bar{q}} \psi + \frac{\cos\psi}{\sin^3\psi}\, \partial_q \varphi\, \partial_{\bar{q}} \varphi + \sin\psi \cos\psi = 0, \\ \partial_{q\bar{q}} \varphi = \frac{1}{\sin\psi \cos\psi}\left(\partial_q \varphi\, \partial_{\bar{q}} \psi + \partial_{\bar{q}} \varphi\, \partial_q \psi\right). \end{cases}$$

Finally, the formula (5.17) and the diagonal part of (5.18) give the second logarithmic derivatives of the τ-function:

$$\begin{cases} \partial_{q\bar{q}} \ln \tau = \cos^2\psi, \\ \partial_{qq} \ln \tau = (\partial_q \psi)^2 + \operatorname{ctg}^2\psi\, (\partial_q \varphi)^2, \\ \partial_{\bar{q}\bar{q}} \ln \tau = (\partial_{\bar{q}} \psi)^2 + \operatorname{ctg}^2\psi\, (\partial_{\bar{q}} \varphi)^2. \end{cases} \tag{5.19}$$

6. Discussion

When one tries to generalize the above theory, a natural question arises: if it is possible to develop the theory of monodromy preserving deformations for the massive Dirac operator on the arbitrary two-dimensional surface M with a metric? It appears that M should be then a homogeneous space for a group G, acting on M by isometries. There are only five such surfaces: plane ($G = E(2)$), cylinder and torus ($G = T^2$), Poincaré disk ($G = PSU(1,1)$) and the sphere ($G = PSU(2)$). The plane and hyperbolic disk were studied earlier by different authors (see references in the Introduction). The present paper is devoted to the cylindrical geometry. It is interesting to note that the derivation of all "implicit" results (factorized form of the derivatives of Green functions, deformation equations, etc.) can be transferred almost literally to the case of torus. What is even more important — the technical results, obtained in this work (namely, the formulae for the one-point projections in the Theorem 4.2) allow to calculate the τ-functions on the torus explicitly. I hope to discuss these matters in greater detail elsewhere.

The second task is to give a proper formulation and solution of the problem in the quantum field theory language. Let us interpret the coordinate along the cylinder axis as time, with the space coordinate living on the circle. The time axis is split by the branchcuts $b_1, \ldots, b_n$ into $n+1$ intervals. The evolution in each interval is governed by the Dirac hamiltonian, which is diagonalized in the free-fermion basis. These free fermions, however, obey different periodicity conditions (statistics) in different intervals. Corresponding Fock spaces are intertwined by the monodromy fields, whose correlation functions can be written in terms of Lehmann expansion over intermediate eigenstates of the hamiltonians. (In the two-point case, this corresponds to the expansion of the determinant (4.25).) The problem transforms then into the calculation of form factors of monodromy fields in the finite volume.

Another important problem is the investigation of the ultraviolet ($m \to 0$) asymptotics of the τ-functions. On the plane, the connection between the Ising model and singular Dirac operators was already used in [11] to give a rigorous proof of the Luther-Peschel formula.

Acknowledgements. I would like to thank A. I. Bugrij and V. N. Roubtsov for constant support and numerous stimulating discussions. I am grateful to S. Pakuliak for his lectures on the infinite-dimensional grassmannians and boson-fermion correspondence. I would also like to express my gratitude to J. Palmer, whose clear ideas made this work possible.

References

1. Bugrij, A.I.: Correlation function of the two-dimensional Ising model on the finite lattice. I. Theor. & Math. Phys. **127**, 528–548 (2001)
2. Bugrij, A.I.: Form factor representation of the correlation functions of the two-dimensional Ising model on a cylinder. In: S. Pakuliak, G. von Gehlen (eds.), *Integrable structures of exactly solvable two-dimensional models of quantum field theory*, NATO Sci. Ser. II Math. Phys. Chem. **35**, Dordrecht Kluwer Acad. Publ. 2001, pp. 65–93
3. Bugrij, A.I., Lisovyy, O.: Spin matrix elements in 2D Ising model on the finite lattice. Phys. Letts. **A319**, 390–394 (2003)
4. Bugrij, A.I., Shadura, V.N.: Asymptotic expression for the correlation function of twisted fields in the two-dimensional Dirac model on a lattice. Theor. & Math. Phys. **121**, 1535–1549 (1999)
5. Doyon, B.: Two-point correlation functions of scaling fields in the Dirac theory on the Poincaré disk. Nucl. Phys. B **675**, 607–630 (2003)
6. Fonseca, P., Zamolodchikov, A.: Ising field theory in a magnetic field: analytic properties of the free energy. J. Stat. Phys. **110**, 527–590 (2003)

7. Lisovyy, O.: Nonlinear differential equations for the correlation functions of the 2D Ising model on the cylinder. Adv. Theor. Math. Phys. **5**, 909–922 (2001)
8. Narayanan, R., Tracy, C.A.: Holonomic quantum field theory of bosons in the Poincaré disk and the zero curvature limit. Nucl. Phys. **B340**, 568–594 (1990)
9. Palmer, J.: Determinants of Cauchy-Riemann operators as τ-functions. Acta Appl. Math. **18**, 199–223 (1990)
10. Palmer, J.: Tau functions for the Dirac operator in the Euclidean plane. Pacific J. Math. **160**, 259–342 (1993)
11. Palmer, J.: *Ising model scaling functions at short distance*. http://arxiv.org/abs/nlin.SI/0107013, 2001
Palmer, J.: *Short distance asymptotics of Ising correlations*. http://arxiv.org/abs/nlin.SI/0107014, 2001
12. Palmer, J., Tracy, C.A.: Monodromy preserving deformation of the Dirac operator acting on the hyperbolic plane. In: M. S. Berger (ed.), *Mathematics of Nonlinear Science: Proceedings of an AMS special session held January 11–14, 1989*, Contemporary Mathematics **108**, Providence, RI: AMS, 1990, pp. 119–131
13. Palmer, J., Beatty, M., Tracy, C.A.: Tau functions for the Dirac operator on the Poincaré disk. Commun. Math. Phys. **165**, 97–173 (1994)
14. Pressley, A., Segal, G.: Loop groups Oxford: Clarendon Press, 1986
15. Quillen, D.: Determinants of Cauchy-Riemann operators on a Riemann surface. Funct. Anal. Appl. **19**, 37–41 (1985)
16. Ray, D.B., Singer, I.M.: Analytic torsion for complex manifolds. Ann. Math. **98**, 154–177 (1973)
17. Sato, M., Miwa, T., Jimbo, M.: *Holonomic quantum fields I–V*. Publ. RIMS, Kyoto Univ. **14**, 223–267 (1978); **15**, 201–278 (1979); **15**, 577–629 (1979); **15**, 871–972 (1979); **16**, 531–584 (1980)
18. Segal, G., Wilson, G.: Loop groups and equations of KdV type. Publ. Math. I.H.E.S. **61**, 5–65 (1985)
19. Witten, E.: Quantum field theory, Grassmannians and algebraic curves. Commun. Math. Phys. **113**, 529–600 (1988)

Communicated by L. Takhtajan