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On Painlevé VI transcendents related to the Dirac operator on the hyperbolic disk

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Dirac Hamiltonian on the Poincaré disk in the presence of an Aharonov–Bohm flux and a uniform magnetic field admits a one-parameter family of self-adjoint extensions. We determine the spectrum and calculate the resolvent for each element of this family. Explicit expressions for Green's functions are then used to find Fredholm determinant representations for the tau function of the Dirac operator with two branch points on the Poincaré disk. Isomonodromic deformation theory for the Dirac equation relates this tau function to a one-parameter class of solutions of the Painlevé VI equation with $\gamma=0$. We analyze long-distance behavior of the tau function, as well as the asymptotics of the corresponding Painlevé VI transcendents as $s\to 1$. Considering the limit of flat space, we also obtain a class of solutions of the Painlevé V equation with $\beta=0$. © *2008 American Institute of Physics.*
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I. INTRODUCTION

It has been known since Refs. 1 and 2 that the two-point correlation function of the two-dimensional (2D) Ising model in the scaling limit is expressible in terms of a solution of a Painlevé III equation. This remarkable result turned out to be a special case of a more general phenomenon, described by Sato–Miwa–Jimbo (SMJ) theory of holonomic quantum fields and monodromy preserving deformations of the Dirac equation.[3] One of the central objects in SMJ theory is the τ-function of the Dirac operator acting on a suitable class of multivalued functions on the Euclidean plane.

The SMJ τ-function admits a geometric interpretation.[4,5] Loosely speaking, it can be obtained by trivializing the det*-bundle over an infinite-dimensional Grassmannian, composed of the spaces of boundary values of certain local solutions of the Dirac equation, where different points of the Grassmannian correspond to different positions of the branch points on the plane. The same idea was used earlier in Ref. 6 to show that the τ-function of the Schlesinger system can be interpreted as a determinant of a singular Cauchy–Riemann operator. A simple finite-dimensional example of this construction arises in the study of a one-dimensional Laplacian with δ-interactions.[7] The most important thing about the geometric picture is that it allows to one find an explicit representation of the SMJ τ-function in terms of a Fredholm determinant, thereby giving a solution of the deformation equations.

The simplest way to generalize the above setup is to replace the plane with an infinite cylinder. In this case, the deformation equations and a Fredholm determinant representation for the τ-function of the corresponding Dirac operator were obtained in Ref. 8. These results provide a shortcut derivation of the partial differential equations satisfied by the scaled Ising correlation functions on the cylinder[9] and of the exact expressions for the one- and two-particle finite-volume form factors of the Ising spin and disorder field[10–12] (and, more generally, of twist fields[13]).

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In Ref. 14, Palmer, Beatty, and Tracy (PBT) extended the SMJ analysis of isomonodromic deformations to the case of a Dirac operator on the Poincaré disk (see also earlier works[15,16] in this subject). The associated τ-function in the simplest nontrivial case of two branch points was shown to be related[14] to a solution of the Painlevé VI (PVI) equation,

$$\frac{d^2w}{ds^2} = \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-s}\right)\left(\frac{dw}{ds}\right)^2 - \left(\frac{1}{s}+\frac{1}{s-1}+\frac{1}{w-s}\right)\frac{dw}{ds} + \frac{w(w-1)(w-s)}{s^2(s-1)^2}\left(\alpha+\frac{\beta s}{w^2} + \gamma\frac{s-1}{(w-1)^2}+\delta\frac{s(s-1)}{(w-s)^2}\right) \tag{1.1}$$

with only one fixed parameter. Before stating the PBT result in more detail, it is useful to reformulate it in a slightly different way. The Dirac operator, considered in Ref. 14, is simply related to the Hamiltonian of a massive Dirac particle moving on the Poincaré disk in the superposition of a uniform magnetic field B and the field of two noninteger Aharonov–Bohm (AB) fluxes $\Phi_{1,2}=2\pi\nu_{1,2}$ located at points a_1 and a_2. Without any loss of generality, one may choose $-1<\nu_{1,2}<0$. It is preferable to work with the Hamiltonian, as it is a symmetric operator that can be made self-adjoint after a proper specification of the domain, and many assertions of Palmer *et al.*[14] (e.g., symmetry of the Green function, nonexistence of certain global solutions of the Dirac equation, etc.) immediately follow from the self-adjointness.

Write the disk curvature as $-4/R^2$, denote by m and E the particle mass and energy, and introduce two dimensionless parameters $b=BR^2/4$ and $\mu=(\sqrt{(m^2-E^2)R^2+4b^2})/2$. It turns out that the τ-function associated to the above Hamiltonian depends only on the geodesic distance $d(a_1,a_2)$ between points a_1 and a_2. If we further introduce $s=\tanh^2(d(a_1,a_2)/R)$, then it can be expressed[14] in terms of a solution of the PVI Eq. (1.1),

$$\frac{d}{ds}\ln\tau(s) = \frac{s(1-s)}{4w(1-w)(w-s)}\left(\frac{dw}{ds}-\frac{1-w}{1-s}\right)^2 - \frac{1-w}{1-s}\left(\frac{\lambda^2}{4s}-\frac{\tilde{\lambda}^2}{4w}+\frac{\mu^2}{w-s}\right), \tag{1.2}$$

where $\lambda=\nu_2-\nu_1$, $\tilde{\lambda}=2+\nu_1+\nu_2-2b$, and the values of the PVI parameters are given by

$$\alpha=\frac{\lambda^2}{2},\quad \beta=-\frac{(\tilde{\lambda}-1)^2}{2},\quad \gamma=0,\quad \delta=\frac{1-4\mu^2}{2}. \tag{1.3}$$

Actually, the paper[14] is concerned with the case $E=0$ (the mass term in the Dirac operator is not of the most general form). We include this parameter from the very beginning because it will be shown below that the final answer for the τ-function depends on E only via the variable μ.

The aim of the present study is to solve the remaining part of the problem, that is, to compute the PBT τ-function and to investigate its asymptotic behavior, which can be used to specify the appropriate initial conditions for Eq. (1.1). We summarize our results in the following theorem:

Theorem 1.1: *The PBT tau function admits Fredholm determinant representation*

$$\tau(s)=\det(\mathbf{1}-L_{\nu_2,s}L'_{\nu_1,s}), \tag{1.4}$$

where the kernels of integral operators $L_{\nu,s}$ and $L'_{\nu,s}$ are

$$L_{\nu,s}(p,q)=e^{i(p-q)l_s/2}\sqrt{\rho(p)\rho(q)}\mathcal{F}_\nu(p,q),$$

$$L'_{\nu,s}(p,q)=L_{\nu,s}(-p,-q),$$

$l_s=\operatorname{arctanh}\sqrt{s}=d(a_1,a_2)/R$ *and* $p,q\in\mathbb{R}$. *The functions* $\rho(p)$ *and* $\mathcal{F}_\nu(p,q)$ *are given by*

$$\rho(p)=\frac{2^{2\mu}\Gamma(1+2\mu)}{\Gamma\left(\mu+\frac{1}{2}+\frac{ip}{2}\right)\Gamma\left(\mu+\frac{1}{2}-\frac{ip}{2}\right)},$$

$$\mathcal{F}_\nu(p,q)=\frac{\sin\pi\nu}{2\pi^2}\int_{-\infty}^{\infty}d\theta\int_0^{\pi/2}dx\int_0^{\pi/2}dy\frac{e^{\{1+\nu+[(1-2b)/2]\}(\theta-2i(x-y))}}{(e^{\theta-2i(x-y)}+1)(2+2\cosh\theta)^{(1+2\mu)/2}}$$
$$\times(\sin x)^{\mu+ip/2-1/2}(\cos x)^{\mu-ip/2-1/2}(\sin y)^{\mu-iq/2-1/2}(\cos y)^{\mu+iq/2-1/2}.$$

To leading order the long-distance $(s\to 1)$ *asymptotics of* $\tau(s)$ *is*

$$1-\tau(s)\simeq A_\tau(1-s)^{1+2\mu}+O((1-s)^{2+2\mu})\quad as\quad s\to 1,$$

where the coefficient A_τ *is given by*

$$A_\tau=\frac{\sin\pi\nu_1\sin\pi\nu_2}{\pi^2}\frac{\Gamma(\mu+2+\nu_1-b)\Gamma(\mu-\nu_1+b)\Gamma(\mu+2+\nu_2-b)\Gamma(\mu-\nu_2+b)}{[\Gamma(2+2\mu)]^2}. \tag{1.5}$$

A few remarks are in order. The formula (1.5) implies that the asymptotic behavior of the corresponding PVI transcendent for $\mu>1/2$ is

$$1-w(s)\simeq A(1-s)^{1+2\mu}+O((1-s)^{2+2\mu})\quad \text{as}\quad s\to 1,$$

$$A=\frac{(1+2\mu)^2}{(\mu+1+\nu_1-b)(\mu+1+\nu_2-b)}A_\tau.$$

The transformation $\nu_1\mapsto\nu_1+c$, $\nu_2\mapsto\nu_2+c$, $b\mapsto b+c$, $\mu\mapsto\mu$ does not change the values (1.3) of the PVI parameters α, β, γ, and δ. However, our solution nontrivially depends on all four variables ν_1, ν_2, b, and μ as can be seen from the above asymptotics. Thus we have constructed a one-parameter family of solutions of the PVI equation with $\gamma=0$.

We also note that in the case $b=0$ the PBT τ-function was conjectured to coincide with a correlation function of $U(1)$ twist fields in the theory of free massive Dirac fermions on the Poincaré disk[17] (particular case of Ising monodromy was later studied in more detail in Refs. 18 and 19). The asymptotics of $\tau(s)$ as $s\to 0$ then follows from the known flat space operator product expansions (OPEs) for twist fields. Long-distance behavior is determined by a form factor expansion of the correlator. The method of angular quantization, employed in Ref. 17 for the calculation of form factors, does not seem to work quite well, as it leads to formally divergent expressions. A sensible answer for the infrared asymptotics was nevertheless extracted from them after a number of regularization procedures. The formula (1.5), specialized to the case $b=0$, proves the latter result.

The main technical problem arising in the direct computation of $\tau(s)$ is the unknown formula for the Green function of the Dirac Hamiltonian on the disk in the presence of a uniform magnetic field and one AB vortex. Such Hamiltonian can always be made commuting with the angular momentum operator by a suitable choice of the gauge. Partial Green's functions are then calculated relatively easily in each channel with fixed angular momentum; the difficult part is the summation of these partial contributions to a closed-form expression. We have solved this problem by writing radial solutions of the Dirac equation as Sommerfeld-type superpositions of horocyclic waves, similar to a simpler scalar case.[20] This allows one to perform the summation and to obtain a simple integral representation for the one-vortex Green's function.

This paper is organized as follows. In Sec. II, after introducing basic notation, we describe the solutions of the radial Dirac equation and compute radial Green's functions. Spectrum, self-adjointness, and admissible boundary conditions for the full Dirac Hamiltonian are also briefly analyzed. Finally, we write contour integral representations for the radial solutions and obtain a compact formula for the one-vortex resolvent. The PBT τ-function is studied in Sec. III. We start by giving a general definition of the τ-function in terms of the projections on some boundary spaces. Next we introduce coordinates in these spaces using the solutions of the Dirac equation on the Poincaré strip. Explicit formulas for the projections in these coordinates are obtained in Sec. III C by analyzing the asymptotics of Green's function, computed in Sec. II. These formulas give the kernels of integral operators in the Fredholm determinant representation of $\tau(s)$. In Sec. III D, we recall the relation of the PBT τ-function to Painlevé VI equation. Section III E deals with the derivation of the long-distance asymptotics of $\tau(s)$. The analogs of the above results in the limit of flat space, where PVI equation transforms into Painlevé V, are established in Sec. IV. The Appendix contains a proof of the fact that the τ-function depends only on the geodesic distance.

II. ONE-VORTEX DIRAC HAMILTONIAN ON THE POINCARÉ DISK

A. Preliminaries

Let us first establish our notations. We denote by D the unit disk $|z|^2<1$ in the complex z-plane, endowed with the Poincaré metric

$$ds^2 = g_{z\bar{z}}dzd\bar{z} = R^2\frac{dzd\bar{z}}{(1-|z|^2)^2}. \tag{2.1}$$

This metric has a constant negative Gaussian curvature $-4/R^2$ and is invariant with respect to the natural $SU(1,1)$-action on D,

$$z \mapsto z_g(z) = \frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}, \quad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1,1). \tag{2.2}$$

The Hamiltonian of a Dirac particle of unit charge moving on the Poincaré disk in an external magnetic field has the form

$$\hat{H} = \begin{pmatrix} m & K \\ K^* & -m \end{pmatrix}, \tag{2.3}$$

where the operator K and its formal adjoint K^* are given by

$$K = \frac{1}{\sqrt{g_{z\bar{z}}}}\left\{2D_z + \frac{1}{2}\partial_z \ln g_{z\bar{z}}\right\}, \tag{2.4}$$

$$K^* = -\frac{1}{\sqrt{g_{z\bar{z}}}}\left\{2D_{\bar{z}} + \frac{1}{2}\partial_{\bar{z}} \ln g_{z\bar{z}}\right\}, \tag{2.5}$$

and $D_z=\partial_z+iA_z$ and $D_{\bar{z}}=\partial_{\bar{z}}+iA_{\bar{z}}$ denote the covariant derivatives.

Connection one-form $\mathcal{A}=A_z dz+A_{\bar{z}}d\bar{z}$, which is considered in the present section, consists of two parts. Namely, we set $\mathcal{A}=\mathcal{A}^{(B)}+\mathcal{A}^{(\nu)}$, where

$$\mathcal{A}^{(B)} = -\frac{iBR^2}{4}\frac{\bar{z}dz - zd\bar{z}}{1-|z|^2}, \tag{2.6}$$

$$\mathcal{A}^{(\nu)} = -\frac{i\nu}{2}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right). \tag{2.7}$$

The first contribution describes a uniform magnetic field of intensity B since $d\mathcal{A}^{(B)}$ is proportional to the volume form $d\mu=(i/2)g_{z\bar{z}}dz\wedge d\bar{z}$. The second part corresponds to the vector potential of an AB flux $\Phi=2\pi\nu$ situated at the disk center.

Introducing polar coordinates $z=re^{i\varphi}$ and $\bar{z}=re^{-i\varphi}$, one can explicitly rewrite the operators K and K^* as follows:

$$K=\frac{e^{-i\varphi}}{R}\left[(1-r^2)\left(\partial_r-\frac{i}{r}\partial_\varphi+\frac{\nu}{r}\right)+(1+2b)r\right], \tag{2.8}$$

$$K^*=-\frac{e^{i\varphi}}{R}\left[(1-r^2)\left(\partial_r+\frac{i}{r}\partial_\varphi-\frac{\nu}{r}\right)+(1-2b)r\right]. \tag{2.9}$$

Here, we have introduced a dimensionless parameter $b=BR^2/4$ characterizing the ratio of magnetic field and the disk curvature.

B. Radial Hamiltonians and self-adjointness

Since the formal Hamiltonian (2.3), corresponding to the vector potential $\mathcal{A}^{(B)}+\mathcal{A}^{(\nu)}$, commutes with the angular momentum operator $\hat{L}=-i\partial_\varphi+\frac{1}{2}\sigma_z$, we will attempt to diagonalize them simultaneously. The eigenvalues of $\hat{L}$ are half-integer numbers $l_0+\frac{1}{2}$ ($l_0\in\mathbb{Z}$) and the appropriate eigenspaces are spanned by the spinors of the form

$$w_{l_0}(r,\varphi)=\begin{pmatrix} w_{l_0,1}(r)e^{il_0\varphi} \\ w_{l_0,2}(r)e^{i(l_0+1)\varphi}\end{pmatrix}.$$

The action of $\hat{H}$ leaves these eigenspaces invariant. One has

$$\begin{pmatrix} w_{l_0,1}(r) \\ w_{l_0,2}(r)\end{pmatrix}\mapsto \hat{H}_{l_0+\nu}\begin{pmatrix} w_{l_0,1}(r) \\ w_{l_0,2}(r)\end{pmatrix},\quad \hat{H}_l=R^{-1}\begin{pmatrix} mR & K_l \\ K_l^* & -mR\end{pmatrix}, \tag{2.10}$$

where the operators K_l and K_l^* are explicitly given by

$$K_l=(1-r^2)\left(\partial_r+\frac{l+1}{r}\right)+(1+2b)r, \tag{2.11}$$

$$K_l^*=-(1-r^2)\left(\partial_r-\frac{l}{r}\right)-(1-2b)r. \tag{2.12}$$

Let us make several remarks concerning the solutions of the radial Dirac equation

$$(\hat{H}_l-E)w_l(r)=0. \tag{2.13}$$

It will be assumed that $E\in\mathbb{C}\setminus((-\infty,-m]\cup[m,\infty))$ and l is an arbitrary real parameter. We also introduce for further convenience the following quantities:

$$\mu=\frac{\sqrt{(m^2-E^2)R^2+4b^2}}{2},$$

$$C_\pm=\left(\frac{m-E}{m+E}\right)^{1/4}\left(\frac{\mu+b}{\mu-b}\right)^{\pm 1/4}.$$

All fractional powers in these formulas are defined so that μ and $C_\pm$ are real and positive for real values of E satisfying $|E|<m$.

Let us first look at the space of solutions of (2.13) on the open unit interval $I=(0,1)$, which are square integrable in the vicinity of the point $r=1$ with respect to the measure $d\mu_r$

$=R^2 r dr/(1-r^2)^2$, induced by the Poincaré metric. It is a simple matter to check that for any $l \in \mathbb{R}$ this space is one dimensional (i.e., the singular point $r=1$ is of the limit point type) and is generated by the function

$$w_l^{(\mathrm{I})}(r) = \frac{\sqrt{\mu^2-b^2}}{2\mu}\frac{\Gamma(\mu-b)\Gamma(\mu+b)}{\Gamma(2\mu)} \times (1-r^2)^{(1+2\mu)/2}\begin{pmatrix} C_+^{-1}r^{-l}{}_2F_1(\mu-b+1,\mu+b-l,1+2\mu,1-r^2) \\ C_+ r^{-l-1}{}_2F_1(\mu-b,\mu+b-l,1+2\mu,1-r^2)\end{pmatrix} \tag{2.14}$$

$$= \frac{\sqrt{\mu^2-b^2}}{2\mu}\frac{\Gamma(\mu-b)\Gamma(\mu+b)}{\Gamma(2\mu)} \times (1-r^2)^{(1+2\mu)/2}\begin{pmatrix} C_+^{-1}r^{l}{}_2F_1(\mu+b,\mu-b+1+l,1+2\mu,1-r^2) \\ C_+ r^{l+1}{}_2F_1(\mu+b+1,\mu-b+1+l,1+2\mu,1-r^2)\end{pmatrix}. \tag{2.15}$$

If $l \in (-\infty,-1] \cup [0,\infty)$, the limit point case is also realized at $r=0$. The solution that satisfies the condition of square integrability in the vicinity of $r=0$ can be written as

$$w_l^{(\mathrm{II},+)}(r) = \begin{pmatrix} \frac{\Gamma(\mu-b+1+l)}{\Gamma(1+l)\Gamma(\mu-b+1)} & 0 \\ 0 & -\frac{\Gamma(\mu-b+1+l)}{\Gamma(2+l)\Gamma(\mu-b)}\end{pmatrix} \times (1-r^2)^{(1+2\mu)/2}\begin{pmatrix} C_+^{-1}r^{l}{}_2F_1(\mu+b,\mu-b+1+l,1+l,r^2) \\ C_+ r^{l+1}{}_2F_1(\mu+b+1,\mu-b+1+l,2+l,r^2)\end{pmatrix}, \tag{2.16}$$

$$w_l^{(\mathrm{II},-)}(r) = \begin{pmatrix} \frac{\Gamma(\mu+b-l)}{\Gamma(1-l)\Gamma(\mu+b)} & 0 \\ 0 & -\frac{\Gamma(\mu+b-l)}{\Gamma(-l)\Gamma(\mu+b+1)}\end{pmatrix} \times (1-r^2)^{(1+2\mu)/2}\begin{pmatrix} C_+^{-1}r^{-l}{}_2F_1(\mu-b+1,\mu+b-l,1-l,r^2) \\ C_+ r^{-l-1}{}_2F_1(\mu-b,\mu+b-l,-l,r^2)\end{pmatrix}, \tag{2.17}$$

where the first formula corresponds to the case $l \geq 0$ and the second one to $l \leq -1$. For $l \in (-1,0)$ both solutions (2.16) and (2.17) are square integrable at $r=0$ (the limit circle case).

The functions $w_l^{(\mathrm{I})}(r)$ and $w_l^{(\mathrm{II},+)}(r)$ are linearly independent for $l > -1$, and the functions $w_l^{(\mathrm{I})}(r)$ and $w_l^{(\mathrm{II},-)}(r)$ are linearly independent for $l < 0$. One can show this, for example, by computing the determinant of the fundamental matrix constructed from these solutions,

$$\det(w_l^{(\mathrm{I})}(r), w_l^{(\mathrm{II},\pm)}(r)) = -\frac{1}{\sqrt{\mu^2-b^2}}\frac{1-r^2}{r}. \tag{2.18}$$

This implies that for $E \in \mathbb{C} \setminus ((-\infty,-m] \cup [m,\infty))$ and $l \in (-\infty,-1] \cup [0,\infty)$ Eq. (2.13) has no square integrable solutions on the whole interval I. For $l \in (-1,0)$, however, there is a one-dimensional space of such solutions, generated by the function (2.14)–(2.15).

We now examine the issue of self-adjointness of the operators $\hat{H}_l$. The above remarks can be summarized as follows.

Proposition 2.1: *Let us restrict the domain of the formal radial Hamiltonian $\hat{H}_l$, defined by* (2.10)–(2.12), *to smooth functions with compact support in I. Then*

- $\hat{H}_l$ *is essentially self-adjoint for* $l \in (-\infty,-1] \cup [0,\infty)$ *and*
- *for* $l \in (-1,0)$, *the operator* $\hat{H}_l$ *has deficiency indices (1,1) and admits a one-parameter family of self-adjoint extensions (SAEs).*

Assume that $l \in (-1,0)$. Deficiency subspaces $\mathcal{K}_{\pm} = \ker(\hat{H}_l^{\dagger} \mp im)$ are generated by the elements

$$w_{\pm}(r) = w_l^{(\mathrm{I})}(r)|_{E=\pm im}. \tag{2.19}$$

Different SAEs $\hat{H}_l^{(\gamma)}$ are in one-to-one correspondence with the isometries between $\mathcal{K}_+$ and $\mathcal{K}_-$. They may be labeled by a parameter $\gamma \in [0,2\pi)$ and characterized by the domains

$$\mathrm{dom}\ \hat{H}_l^{(\gamma)} = \{w_0 + c(w_+ + e^{i\gamma} w_-) | w_0 \in \mathrm{dom}\ \hat{H}_l, c \in \mathbb{C}\}. \tag{2.20}$$

It is also conventional to characterize the functions from the domain of the closure of $\hat{H}_l^{(\gamma)}$ by their asymptotic behavior near the point $r=0$. Namely, it follows from (2.15), (2.19), and (2.20) that for $w \in \mathrm{dom}(\overline{\hat{H}_l^{(\gamma)}})$, one should have

$$\lim_{r\to 0} \cos\left(\frac{\Theta}{2} + \frac{\pi}{4}\right)(mRr)^{-l} w_1(r) = -\lim_{r\to 0} \sin\left(\frac{\Theta}{2} + \frac{\pi}{4}\right)(mRr)^{1+l} w_2(r). \tag{2.21}$$

Here, we have introduced instead of γ a new SAE parameter $\Theta \in [0,2\pi)$, defined by

$$\tan\left(\frac{\Theta}{2} + \frac{\pi}{4}\right) = \frac{2^{-l}}{\tan\left(\frac{\gamma}{2} - \frac{\pi}{8}\right) - 1} \frac{\Gamma(-l)}{\Gamma(1+l)} \frac{\Gamma(\tilde{\mu}+b+1)\Gamma(\tilde{\mu}-b+l+1)}{\Gamma(\tilde{\mu}-b+1)\Gamma(\tilde{\mu}+b-l)} \sqrt{\frac{\tilde{\mu}-b}{\tilde{\mu}+b}} \left(\frac{mR}{\sqrt{2}}\right)^{-1-2l}, \tag{2.22}$$

where $\tilde{\mu} = \mu|_{E=\pm im} = (\sqrt{2m^2R^2 + 4b^2})/2$. Note that the choice $\Theta = \pi/2 (\Theta = -\pi/2)$ is equivalent to requiring the regularity of the lower (upper) component of the Dirac spinor at $r=0$.

Let us now consider full Dirac Hamiltonian. Since the shift of the AB flux by any integer number is equivalent to a unitary transformation of $\hat{H}$, hereafter we will assume that $-1 < \nu \leq 0$.

Proposition 2.2: *Suppose that* dom $\hat{H} = C_0^{\infty}(D \backslash \{0\})$. *Then*

- *for* $\nu=0$ *the operator* $\hat{H}$ *is essentially self-adjoint and*
- *for* $\nu \in (-1,0)$, *it has deficiency indices (1,1) and admits a one-parameter family of SAEs, henceforth denoted by* $\hat{H}^{(\gamma)}$, *which correspond to those of the radial mode with* $l_0 = 0$.

C. Radial Green's functions

Let us begin with the case $l \in (-\infty,-1] \cup [0,\infty)$. Green's function $G_{E,l}(r,r')$ of the radial Hamiltonian $\hat{H}_l$ can be viewed as the solution of the equation

$$(\hat{H}_l(r) - E) G_{E,l}(r,r') = \frac{(1-r^2)^2}{R^2 r} \delta(r-r') \mathbf{1}_2, \tag{2.23}$$

which is square integrable in the vicinity of the boundary points $r=0$ and $r=1$. Standard ansatz

$$G_{E,l}(r,r') = \begin{cases} A_{E,l}^{\pm} w_l^{(\mathrm{II},\pm)}(r) \otimes (w_l^{(\mathrm{I})}(r'))^T & \text{for } 0 < r < r' < 1 \\ A_{E,l}^{\pm} w_l^{(\mathrm{I})}(r) \otimes (w_l^{(\mathrm{II},\pm)}(r'))^T & \text{for } 0 < r' < r < 1 \end{cases} \tag{2.24}$$

solves (2.23) for $r \neq r'$ and meets the requirements of square integrability. The sign "+" ("−") in the above formula should be chosen for $l \geq 0$ ($l \leq -1$). Prescribed singular behavior of the Green's function at the point $r=r'$ is equivalent to the condition

$$G_{E,l}(r+0,r) - G_{E,l}(r-0,r) = -\frac{i}{R}\frac{1-r^2}{r}\sigma_y.$$

Substituting (2.24) into the last relation, we may rewrite it as follows:

$$A^{\pm}_{E,l} \det W(w_l^{(\mathrm{I})}(r), w_l^{(\mathrm{II},\pm)}(r)) = -\frac{1-r^2}{Rr}.$$

Finally, using (2.18) one finds that

$$A^{+}_{E,l} = A^{-}_{E,l} = \frac{\sqrt{m^2-E^2}}{2}.$$

Now suppose that $l \in (-1,0)$. In order to find the resolvent of the radial Hamiltonian $\hat{H}_l^{(\gamma)}$, we need a solution of the radial Dirac equation $(\hat{H}_l^{(\gamma)}-E)w_l^{(\gamma)}=0$, which satisfies the boundary condition (2.21) at the point $r=0$ (square integrability near the point $r=1$ is not required). Such a solution can always be represented as a linear combination of the functions $w_l^{(\mathrm{II},\pm)}(r)$ defined by (2.16) and (2.17). These functions have the following asymptotic behavior as $r\to 0$:

$$w_l^{(\mathrm{II},+)}(r) = \frac{\Gamma(\mu-b+1+l)}{\Gamma(1+l)\Gamma(\mu-b+1)}\begin{pmatrix} C_+^{-1}r^l \\ 0 \end{pmatrix} + O(r^{1+l}), \tag{2.25}$$

$$w_l^{(\mathrm{II},-)}(r) = -\frac{\Gamma(\mu+b-l)}{\Gamma(-l)\Gamma(\mu+b+1)}\begin{pmatrix} 0 \\ C_+ r^{-l-1} \end{pmatrix} + O(r^{-l}). \tag{2.26}$$

Therefore, the solution $w_l^{(\gamma)}(r)$ can be written as

$$w_l^{(\gamma)}(r) = \cos\,\eta w_l^{(\mathrm{II},+)}(r) + \sin\,\eta w_l^{(\mathrm{II},-)}(r), \tag{2.27}$$

Note that for special values of SAE parameter, $\Theta=(\pi/2)(\Theta=-\pi/2)$ we have $\eta=0$ $(\eta=\pi/2)$. Explicit dependence of η on Θ, l, μ, and b in the general case can be easily found from (2.21), (2.25), and (2.26). Now, analogously to the above, consider the following ansatz for the Green's function:

$$G_{E,l}^{(\gamma)}(r,r') = \begin{cases} A_{E,l}^{(\gamma)} w_l^{(\gamma)}(r) \otimes (w_l^{(\mathrm{I})}(r'))^T & \text{for } 0<r<r'<1 \\ A_{E,l}^{(\gamma)} w_l^{(\mathrm{I})}(r) \otimes (w_l^{(\gamma)}(r'))^T & \text{for } 0<r'<r<1. \end{cases} \tag{2.28}$$

This ansatz automatically solves Eq. (2.23) for $r\neq r'$ and satisfies the appropriate boundary conditions at the points $r=0$ and $r=1$. The jump condition at $r=r'$ will be satisfied provided we have

$$A_{E,l}^{(\gamma)} \det W(w_l^{(\mathrm{I})}(r), w_l^{(\gamma)}(r)) = -\frac{1-r^2}{Rr}.$$

The last condition trivially holds if one chooses

$$A_{E,l}^{(\gamma)} = \frac{\sqrt{m^2-E^2}}{2}\frac{1}{\sqrt{2}\sin\left(\eta+\frac{\pi}{4}\right)}. \tag{2.29}$$

Hence the formulas (2.27)–(2.29) give the radial Green's function for $l\in(-1,0)$.

D. Spectrum

The spectrum of the full Dirac Hamiltonian $\hat{H}^{(\gamma)}$ consists of several parts.

- a continuous spectrum $|E|^2 \geq m^2+4b^2/R^2$;

- a finite number of infinitely degenerate Landau levels, given by

$$|E_n^{(0)}|^2 = m^2 + \frac{4}{R^2}[b^2 - (|b| - n)^2],$$

where $n=1,2,\ldots,n_{\max}<|b|$ (the allowed eigenvalues of angular momentum correspond to $l_0=-1,-2,\ldots$ for $b>0$ and $l_0=1,2,\ldots$ for $b<0$);
- a finite number of bound states with finite degeneracy, whose form depends on the sign of the magnetic field, namely, for $b>0$ one has

$$|E_n^{(\nu,+)}|^2 = m^2 + \frac{4}{R^2}[b^2 - (b - n - (1+\nu))^2],$$

where $n=1,2,\ldots,n'_{\max}<b-(1+\nu)$, and for $b<0$ we obtain

$$|E_n^{(\nu,-)}|^2 = m^2 + \frac{4}{R^2}[b^2 - (|b| - n + \nu)^2]$$

with $n=1,2,\ldots,n''_{\max}<|b|+\nu$ [the allowed angular momenta are given by $l_0=1,2,\ldots,n'_{\max}$ (for $b>0$) and $l_0=-1,-2,\ldots,-n''_{\max}$ (for $b<0$)]; and
- a finite number of nondegenerate bound states, corresponding to the mode with $l_0=0$. These energy levels are determined as real roots of the equation

$$\frac{(m+E)R}{2}\left(\frac{2}{mR}\right)^{1+2\nu}\frac{\Gamma(\mu+b)\Gamma(\mu-b+\nu+1)}{\Gamma(\mu-b+1)\Gamma(\mu+b-\nu)} = -\frac{\Gamma(\nu+1)}{\Gamma(-\nu)}2^{1+2\nu}\tan\left(\frac{\Theta}{2}+\frac{\pi}{4}\right) \overset{\text{def}}{=} -A(\Theta,\nu).$$

Note that for $A(\Theta,\nu)<0$ we can have a solution of this equation satisfying $|E|<m$. This is in constrast with the previous cases, where all energy levels lie in the interval $m^2<E^2<m^2+4b^2/R^2$.

E. Full one-vortex Green's function

Once the Green's function of a particular SAE is found, one can also obtain it for any other extension using Krein's formula. This fact very much simplifies the analysis of the δ-interaction Hamiltonians (see, e.g., Ref. 21) since in that case the family of SAEs usually includes free Laplacian/Dirac operator, whose Green's function can be computed relatively easily (for example, in the planar case one has just to apply Fourier transform). The situation with AB Hamiltonians is different. Here, the calculation of the resolvent constitutes a nontrivial problem even for distinguished values of the extension parameters.

In the present subsection, we obtain integral representations for the Green's function

$$G(z,z') = \frac{1}{2\pi}\sum_{l_0\in\mathbb{Z}}\begin{pmatrix} e^{il_0\varphi} & 0 \\ 0 & e^{i(l_0+1)\varphi}\end{pmatrix} G_{E,l_0+\nu}(r,r')\begin{pmatrix} e^{-il_0\varphi'} & 0 \\ 0 & e^{-i(l_0+1)\varphi'}\end{pmatrix} \tag{2.30}$$

of the full Hamiltonian $H^{(\gamma)}$ for two values of SAE parameter, namely, for $\Theta=\pm\pi/2$. The outline of the calculation is similar to Ref. 20 and the reader is referred to this paper for more details.

We begin by introducing two classes of solutions of the Dirac equation on the disk without AB field,

$$\Psi_\pm(z,\theta)=\begin{pmatrix} C_\pm^{-1}e^{-\theta/2}\dfrac{(1-|z|^2)^{(1\pm2\mu)/2}}{(1+ze^{-\theta})^{1\pm\mu-b}(1+\bar{z}e^{\theta})^{\pm\mu+b}} \\ \pm C_\pm e^{\theta/2}\dfrac{(1-|z|^2)^{(1\pm2\mu)/2}}{(1+ze^{-\theta})^{\pm\mu-b}(1+\bar{z}e^{\theta})^{1\pm\mu+b}} \end{pmatrix}.$$

These functions are delimited by two families of branch cuts in the θ-plane: $[-\infty+i(\varphi+\pi+2\pi\mathbb{Z}),\ln r+i(\varphi+\pi+2\pi\mathbb{Z})]$ and $[-\ln r+i(\varphi+\pi+2\pi\mathbb{Z}),\infty+i(\varphi+\pi+2\pi\mathbb{Z})]$, with the arguments of $1+ze^{-\theta}$ and $1+\bar{z}e^{\theta}$ equal to zero on the line $\text{Im }\theta=\varphi$. It is also convenient to introduce the "conjugates" of these solutions, defined by

$$\hat{\Psi}_\pm(z,\theta)=\Psi_\pm(z\leftrightarrow\bar{z},\theta\leftrightarrow-\theta). \tag{2.31}$$

The relation $\hat{L}(z)\Psi_\pm(z,\theta)=-\partial_\theta\Psi_\pm(z,\theta)$ allows one to construct multivalued radial solutions of the Dirac equation with specified monodromy as superpositions of $\Psi_\pm(z,\theta)$. One can check that

$$\mathrm{w}_l^{(\mathrm{I})}(z)\overset{\text{def}}{=}\int_{C_0(z)}e^{(l+1/2)\theta}\Psi_-(z,\theta)d\theta=e^{i\pi l}\begin{pmatrix} e^{il\varphi} & 0\\ 0 & e^{i(l+1)\varphi}\end{pmatrix}w_l^{(\mathrm{I})}(r), \tag{2.32}$$

$$\mathrm{w}_l^{(\mathrm{II},\pm)}(z)\overset{\text{def}}{=}\pm\int_{C_\pm(z)}e^{(l+1/2)\theta}\Psi_+(z,\theta)d\theta=2\pi ie^{i\pi l}\begin{pmatrix} e^{il\varphi} & 0\\ 0 & e^{i(l+1)\varphi}\end{pmatrix}w_l^{(\mathrm{II},\pm)}(r), \tag{2.33}$$

where the contour $C_+(z)$ $(C_-(z))$ goes counterclockwise around the branch cut $[-\infty+i(\varphi+\pi),\ln r+i(\varphi+\pi)]$ $([-\ln r+i(\varphi+\pi),\infty+i(\varphi+\pi)])$, and the contour $C_0(z)$ is the line segment joining the branch points $\pm\ln r+i(\varphi+\pi)$. Conjugate solutions are obtained analogously

$$\hat{\mathrm{w}}_l^{(\mathrm{I})}(z)\overset{\text{def}}{=}\int_{C_0(z)}e^{-(l+1/2)\theta}\hat{\Psi}_-(z,\theta)d\theta=e^{-i\pi l}\begin{pmatrix} e^{-il\varphi} & 0\\ 0 & e^{-i(l+1)\varphi}\end{pmatrix}w_l^{(\mathrm{I})}(r), \tag{2.34}$$

$$\hat{\mathrm{w}}_l^{(\mathrm{II},\pm)}(z)\overset{\text{def}}{=}\mp\int_{C_\mp(z)}e^{-(l+1/2)\theta}\hat{\Psi}_+(z,\theta)d\theta=2\pi ie^{-i\pi l}\begin{pmatrix} e^{-il\varphi} & 0\\ 0 & e^{-i(l+1)\varphi}\end{pmatrix}w_l^{(\mathrm{II},\pm)}(r). \tag{2.35}$$

Let us assume the regularity of the upper component of the Dirac wave function, i.e., $\Theta=-\pi/2$. Then Green's function (2.30) can be conveniently expressed in terms of the radial solutions (2.32)–(2.35) as follows:

$$G(z,z')=\frac{\sqrt{m^2-E^2}}{8i\pi^2}e^{-i\nu(\varphi-\varphi')}[\mathcal{G}^{(+)}(z,z')+\mathcal{G}^{(-)}(z,z')], \tag{2.36}$$

where

$$\mathcal{G}^{(\pm)}(z,z')=\sum_{l\in\mathbb{Z}+\nu,l\gtrless0}\mathrm{w}_l^{(\mathrm{I})}(z)\otimes(\hat{\mathrm{w}}_l^{(\mathrm{II},\pm)}(z'))^T\quad\text{for}\quad|z|>|z'|, \tag{2.37}$$

$$\mathcal{G}^{(\pm)}(z,z')=\sum_{l\in\mathbb{Z}+\nu,l\gtrless0}\mathrm{w}_l^{(\mathrm{II},\pm)}(z)\otimes(\hat{\mathrm{w}}_l^{(\mathrm{I})}(z'))^T\quad\text{for}\quad|z|<|z'|. \tag{2.38}$$

Remark: For $\Theta=\pi/2$, one obtains similar representations, but in this case the summation in $\mathcal{G}^{(\pm)}(z,z')$ is over $l\gtrless-1$. Further calculation is also completely analogous, so we will continue with $\Theta=-\pi/2$, and present only the final result for $\Theta=\pi/2$ at the end of this section.

Following Ref. 20, we substitute the contour integral representations (2.32)–(2.35) into (2.37) and (2.38) instead of $\mathrm{w}_l^{(I)}(z)$, $\hat{\mathrm{w}}_l^{(\mathrm{I})}(z')$, $\mathrm{w}_l^{(\mathrm{II},\pm)}(z)$, and $\hat{\mathrm{w}}_l^{(\mathrm{II},\pm)}(z')$. After interchanging the order of summation and integration, the sums over l are reduced to geometric series. For example, in the case $|z|>|z'|$ this gives

$$\mathcal{G}_k^{(+)}(z,z') + \mathcal{G}_k^{(-)}(z,z') = \int_{C_0(z)} d\theta_1 \int_{C_+(z')\cup C_-(z')} d\theta_2 \;\; \Psi_-(z,\theta_1) \otimes \hat{\Psi}_+^T(z',\theta_2) \frac{e^{(1+\nu+1/2)(\theta_1-\theta_2)}}{e^{\theta_1-\theta_2}-1}, \tag{2.39}$$

where the contours $C_\pm(z')$ satisfy additional constraints: $\mathrm{Re}(\theta_1-\theta_2)<0$ for all $\theta_1 \in C_0(z)$, $\theta_2 \in C_-(z')$ and $\mathrm{Re}(\theta_1-\theta_2)>0$ for all $\theta_1 \in C_0(z)$, $\theta_2 \in C_+(z')$. After a suitable deformation of integration contours, one can obtain a representation, which is valid not only for $|z|>|z'|$ but also for all z and z' such that $\varphi-\varphi' \neq \pm\pi$. It has the following form [cf. with (2.24)–(2.26) in Ref. 4]:

$$G(z,z') = \begin{cases} e^{-i\nu(\varphi-\varphi'+2\pi)}G^{(0)}(z,z') + \Delta(z,z') & \text{for} \quad \varphi-\varphi' \in (-2\pi,-\pi) \\ e^{-i\nu(\varphi-\varphi')}G^{(0)}(z,z') + \Delta(z,z') & \text{for} \quad \varphi-\varphi' \in (-\pi,\pi) \\ e^{-i\nu(\varphi-\varphi'-2\pi)}G^{(0)}(z,z') + \Delta(z,z') & \text{for} \quad \varphi-\varphi' \in (\pi,2\pi), \end{cases} \tag{2.40}$$

with

$$G^{(0)}(z,z') = \frac{\sqrt{m^2-E^2}}{4\pi}\int_{C_0(z)} d\theta \;\; \Psi_-(z,\theta) \otimes \hat{\Psi}_+^T(z',\theta), \tag{2.41}$$

$$\Delta(z,z') = \sqrt{m^2-E^2}\,e^{-i\nu(\varphi-\varphi')}\frac{1-e^{-2\pi i\nu}}{8i\pi^2}\int_{C_0(z)} d\theta_1 \int_{\mathrm{Im}\,\theta_2=\varphi'} d\theta_2 \;\; \Psi_-(z,\theta_1)$$
$$\otimes\, \hat{\Psi}_+^T(z',\theta_2)\frac{e^{[1+\nu+(1/2)](\theta_1-\theta_2)}}{e^{\theta_1-\theta_2}-1}. \tag{2.42}$$

Remark: Starting from (2.38), similar results can be obtained. More precisely, one finds again the formula (2.40) but with

$$G^{(0)}(z,z') = \frac{\sqrt{m^2-E^2}}{4\pi}\int_{C_0(z')} d\theta \;\; \Psi_+(z,\theta) \otimes \hat{\Psi}_-^T(z',\theta), \tag{2.43}$$

$$\Delta(z,z') = \sqrt{m^2-E^2}\,e^{-i\nu(\varphi-\varphi')}\frac{1-e^{2\pi i\nu}}{8i\pi^2}\int_{C_0(z')} d\theta_1 \int_{\mathrm{Im}\,\theta_2=\varphi} d\theta_2 \;\; \Psi_+(z,\theta_2) \otimes \hat{\Psi}_-^T(z',\theta_1)\frac{e^{(1+\nu+1/2)(\theta_2-\theta_1)}}{1-e^{\theta_2-\theta_1}}. \tag{2.44}$$

The proof of equivalence of the representations (2.41)–(2.44) for $G^{(0)}(z,z')$ and $\Delta(z,z')$ is left to the reader as an exercise.

Since for $\nu=0$ the second term in (2.40) vanishes, $G^{(0)}(z,z')$ coincides with the Green's function of the Dirac Hamiltonian without AB field, which we will denote by $\hat{H}^{(0)}$. Using the technique described in the Appendix A of Ref. 20, one may compute the integral for $G^{(0)}(z,z')$ in terms of hypergeometric functions. The result reads as

$$G^{(0)}(z,z') = \left(\frac{1-\bar{z}z'}{1-z\bar{z}'}\right)^{-b}\begin{pmatrix} \left(\frac{1-\bar{z}z'}{1-z\bar{z}'}\right)^{1/2}\zeta_{11}(u(z,z')) & \frac{|1-\bar{z}z'|}{z'-z}\zeta_{12}(u(z,z')) \\ -\frac{|1-\bar{z}z'|}{\bar{z}'-\bar{z}}\zeta_{21}(u(z,z')) & -\left(\frac{1-\bar{z}z'}{1-z\bar{z}'}\right)^{-1/2}\zeta_{22}(u(z,z')) \end{pmatrix}, \tag{2.45}$$

where $u(z,z')=|(z'-z)/(1-\bar{z}z')|^2$ and

$$\zeta(u) = \frac{1}{2\pi R}\frac{\Gamma(\mu-b+1)\Gamma(\mu+b+1)}{\Gamma(1+2\mu)}(1-u)^{(1+2\mu)/2} \times \begin{pmatrix} C_+^{-2}{}_2F_1(\mu-b+1,\mu+b,1+2\mu,1-u) & {}_2F_1(\mu-b,\mu+b,1+2\mu,1-u) \\ {}_2F_1(\mu-b,\mu+b,1+2\mu,1-u) & C_+^2{}_2F_1(\mu-b,\mu+b+1,1+2\mu,1-u) \end{pmatrix}. \tag{2.46}$$

One may also obtain a more explicit expression for $\Delta(z,z')$,

$$\Delta(z,z') = \frac{\sin \pi\nu}{\pi}\int_{-\infty}^{\infty} d\theta \frac{e^{(1+\nu)\theta+i(\varphi-\varphi')}}{e^{\theta+i(\varphi-\varphi')}+1}\left(\frac{1+rr'e^{\theta}}{1+rr'e^{-\theta}}\right)^{-b} \times \begin{pmatrix} \left(\frac{1+rr'e^{\theta}}{1+rr'e^{-\theta}}\right)^{1/2}\zeta_{11}(v(r,r',\theta)) & \frac{(1+r^2r'^2+2rr'\cosh\theta)^{1/2}}{re^{-\theta}+r'}e^{-i\varphi'}\zeta_{12}(v(r,r',\theta)) \\ \frac{(1+r^2r'^2+2rr'\cosh\theta)^{1/2}}{re^{\theta}+r'}e^{\theta+i\varphi}\zeta_{21}(v(r,r',\theta)) & e^{\theta+i(\varphi-\varphi')}\left(\frac{1+rr'e^{\theta}}{1+rr'e^{-\theta}}\right)^{-1/2}\zeta_{22}(v(r,r',\theta)) \end{pmatrix}, \tag{2.47}$$

where we have introduced the notation

$$v(r,r',\theta) = \frac{r^2+r'^2+2rr'\cosh\theta}{1+r^2r'^2+2rr'\cosh\theta}. \tag{2.48}$$

Remark: In the case $\Theta=\pi/2$, one finds essentially the same answer (2.40). The only difference is that we should replace ν with $\nu-1$ in the double integrals entering the definitions (2.42) and (2.44) of $\Delta(z,z')$. This finally gives

$$\Delta(z,z') = \frac{\sin \pi\nu}{\pi}\int_{-\infty}^{\infty} d\theta \frac{-e^{\nu\theta}}{e^{\theta+i(\varphi-\varphi')}+1}\left(\frac{1+rr'e^{\theta}}{1+rr'e^{-\theta}}\right)^{-b} \times \begin{pmatrix} \left(\frac{1+rr'e^{\theta}}{1+rr'e^{-\theta}}\right)^{1/2}\zeta_{11}(v(r,r',\theta)) & \frac{(1+r^2r'^2+2rr'\cosh\theta)^{1/2}}{re^{-\theta}+r'}e^{-i\varphi'}\zeta_{12}(v(r,r',\theta)) \\ \frac{(1+r^2r'^2+2rr'\cosh\theta)^{1/2}}{re^{\theta}+r'}e^{\theta+i\varphi}\zeta_{21}(v(r,r',\theta)) & e^{\theta+i(\varphi-\varphi')}\left(\frac{1+rr'e^{\theta}}{1+rr'e^{-\theta}}\right)^{-1/2}\zeta_{22}(v(r,r',\theta)) \end{pmatrix}. \tag{2.49}$$

Representations (2.47) and (2.49) for the vortex-dependent part of Green's function constitute the main technical result of this section, which will be used later in the construction of PVI transcendents.

III. TWO-POINT TAU FUNCTION

A. General setting

In this section, we consider Dirac Hamiltonian (2.3) with two AB vortices located at points $a_1, a_2 \in D$. The corresponding vector potential has the form

$$\mathcal{A} = -\frac{iBR^2}{4}\frac{\bar{z}dz - zd\bar{z}}{1-|z|^2} - \frac{i\nu_1}{2}\left(\frac{dz}{z-a_1} - \frac{d\bar{z}}{\bar{z}-\bar{a}_1}\right) - \frac{i\nu_2}{2}\left(\frac{dz}{z-a_2} - \frac{d\bar{z}}{\bar{z}-\bar{a}_2}\right), \tag{3.1}$$

and it is assumed that $-1<\nu_{1,2}<0$. Let us make a singular gauge transformation $\hat{H}\mapsto\hat{H}^{(a,\nu)} = U\hat{H}U^{\dagger}$ with

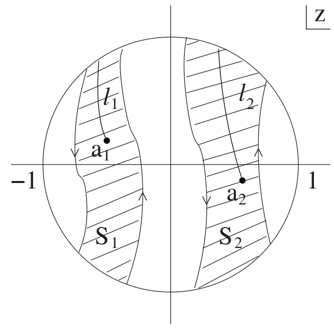

FIG. 1. Branch cuts and localization domains on the Poincaré disk.

$$(U\psi)(z) = \left(\frac{z-a_1}{\bar{z}-\bar{a}_1}\right)^{\nu_1/2}\left(\frac{z-a_2}{\bar{z}-\bar{a}_2}\right)^{\nu_2/2}\psi(z). \tag{3.2}$$

It is easy to check that the local action of $\hat{H}^{(a,\nu)}$ coincides with that of the free Hamiltonian $\hat{H}^{(0)}$ (i.e., in the absence of AB fluxes). However, the functions from the domain of $\hat{H}^{(a,\nu)}$ are multivalued: they pick up a phase given by $e^{2\pi i\nu_j}$ when continued around a_j ($j=1,2$). One should then introduce two branch cuts ℓ_1 and ℓ_2 on D, as shown in Fig. 1. We do not fix the branches of fractional powers in (3.2) since this will be implicitly done later.

Let us isolate the branch cuts in the union S of two open strips S_1 and S_2 (see Fig. 1) and denote by $\mathcal{H}(S)$ the set of all Hamiltonians $\hat{H}^{(a,\nu)}$ satisfying $a_1 \in S_1$ and $a_2 \in S_2$. In other words, the elements of $\mathcal{H}(S)$ are parametrized by the positions of AB vortices, their fluxes being fixed. Consider the localizations of different elements of $\mathcal{H}(S)$ to $D\backslash S$. Since all of them are given by the same differential operator, all dependence on a is encoded into the spaces of boundary values on ∂S of the functions from their domains. Several such boundary spaces will be considered:

- a suitably chosen space $W=H^{1/2}(\partial S)$ of $\mathbb{C}^2$-valued functions on ∂S;
- a subspace $W^{\text{int}}(a)\subset W$, which is composed of boundary values of functions $\psi \in H^1(S\backslash(\ell_1\cup\ell_2))$ solving the equation $(\hat{H}^{(a,\nu)}-E)\psi=0$ on S [note that $W^{\text{int}}(a)=W_1^{\text{int}}(a_1)\oplus W_2^{\text{int}}(a_2)$, where $W_j^{\text{int}}(a_j)$ ($j=1,2$) is composed of the boundary values on ∂S_j of local solutions to the Dirac equation on the strip S_j];
- similarly, $W^{\text{ext}}\subset W$ is defined to be the space of boundary values of H^1-solutions of $(\hat{H}^{(a,\nu)}-E)\psi=0$ on $D\backslash\bar{S}$ (this subspace clearly does not depend on a); and
- it is also convenient to fix two points $a_1^0\in S_1$ and $a_2^0 \in S_2$ and to introduce a reference subspace $W^{\text{int}}(a^0)$.

It will be assumed that near each branch point, one of the components of the Dirac spinor is regular and the other is square integrable (i.e., we consider four possible types of boundary conditions for $\hat{H}^{(a,\nu)}$). With this choice of the domain, the spaces $W^{\text{int}}(a)$ and W^{ext} can be shown to be transverse in W for $E\in\mathbb{C}\backslash((-\infty,-m]\cup[m,\infty))$. Then, given any subspace $V\subset W$, one can define the projection $P(a):V\xrightarrow{W^{\text{ext}}} W^{\text{int}}(a)$ of V on $W^{\text{int}}(a)$ along W^{ext}.

Proposition 3.1: *Let $f\in W$ and consider the function $f_{\text{int}}\in H^1(S\backslash(\ell_1\cup\ell_2))$ defined by*

$$f_{\text{int}}(z) = -\int_{\partial S}\ddot{G}^{(a,\nu)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_- f(z')dz' + \sigma_+ f(z')d\bar{z}'\},$$

$$\sigma_+ = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad \sigma_- = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \tag{3.3}$$

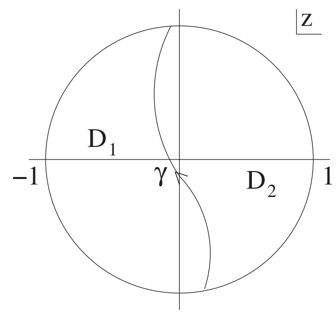

FIG. 2. Orientation of γ.

where $z\in S\backslash(\ell_1\cup\ell_2)$, ∂S is oriented counterclockwise, and $\ddot{G}^{(a,\nu)}(z,z')$ denotes the Green's function of $\hat{H}^{(a,\nu)}$. Then the boundary value of f_{int} on ∂S coincides with the projection $P(a)f\in W^{\text{int}}(a)$.

First remark that f_{int} satisfies Dirac equation on S. Therefore, we may write

$$f_{\text{int}}(z)=\int_{D\backslash\bar{S}}\ddot{G}^{(a,\nu)}(z,z')(\hat{H}^{(a,\nu)}-E)f_{\text{int}}(z')d\mu_{z'}.$$

Integrating once by parts and using Stokes theorem, one obtains

$$f_{\text{int}}(z)=-\int_{\partial S}\ddot{G}^{(a,\nu)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_-f_{\text{int}}(z')dz'+\sigma_+f_{\text{int}}(z')d\bar{z}'\},\tag{3.4}$$

and thus the map (3.3) is indeed a projection on $W^{\text{int}}(a)$. Using similar arguments, one can show that its kernel coincides with W^{ext}. □

Definition 3.2: *Let $f_j\in H^{1/2}(\partial S_j)$ $(j=1,2)$ and consider the function $f_{\text{int},j}\in H^1(S_j\backslash\ell_j)$ defined by*

$$f_{\text{int},j}(z)=-\int_{\partial S_j}\dot{G}^{(a_j,\nu_j)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_-f_j(z')dz'+\sigma_+f_j(z')d\bar{z}'\},\tag{3.5}$$

where $z\in S_j\backslash\ell_j$ and $\dot{G}^{(a_j,\nu_j)}(z,z')$ is the Green's function of the Dirac Hamiltonian on the disk with only one branch point a_j. Passing to boundary values of $f_{\text{int},j}$ on ∂S_j, one obtains a projection $P_j(a_j):H^{1/2}(\partial S_j)\to W_j^{\text{int}}(a_j)$. We will denote by $F(a)$ the direct sum of such one-point projections:

$$F(a)\overset{\text{def}}{=}P_1(a_1)\oplus P_2(a_2):W\to W^{\text{int}}(a).$$

Definition 3.3: *τ-function of $\hat{H}^{(a,\nu)}$ is defined as follows:*

$$\tau(a,a^0)=\det([P_1(a_1)\oplus P_2(a_2)]_{W^{\text{int}}(a^0)}\overset{W^{\text{ext}}}{\to}W^{\text{int}}(a^0))=\det_{W^{\text{int}}(a^0)}(P^{-1}(a)F(a)).\tag{3.6}$$

Remark: For this definition to make sense, the restriction of the map $P^{-1}(a)F(a)$ to $W^{\text{int}}(a^0)$ should be a trace class perturbation of the identity on $W^{\text{int}}(a^0)$. This can be shown analogously to Proposition 4.2 in Ref. 4. Factorization formulas for the derivatives of one-point Green's functions $\dot{G}^{(a_j,\nu_j)}(z,z')$, needed for the proof, are presented in Sec. III D 3.

We are interested in explicit calculation of the τ-function (3.6). In order to do this, it is convenient to introduce yet other boundary maps.

Definition 3.4: *Let γ be a smooth curve dividing Poincaré disk into two disconnected parts D_1 and D_2 as shown in Fig. 2. For any $f\in H^{1/2}(\gamma)$ define the functions*

$$f_{\pm}(z) = \pm \int_{\gamma} G^{(0)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_- f(z')dz' + \sigma_+ f(z')d\bar{z}'\}, \tag{3.7}$$

where $z \in D_2$ for $f_+(z)$, $z \in D_1$ for $f_-(z)$, and $G^{(0)}(z,z')$ is the Green's function of the Dirac Hamiltonian without AB field, explicitly given by (2.45) and (2.46). With some abuse of notation, the boundary values of $f_{\pm}(z)$ on γ will also be denoted by $f_{\pm}(z)$.

Note that $f_{\pm}$ have the properties of projections

$$(f_{\pm})_{\pm} = f_{\pm} \quad (f_{\pm})_{\mp} = 0$$

and, in addition, $f_+ + f_- = f$. One can thus uniquely write any $f \in H^{1/2}(\gamma)$ as a sum of two functions, where the first term (f_+) may be continued from γ to D_2 as a solution of the Dirac equation without AB field, and the second one (f_-) is the boundary value of a solution on D_1.

Let us represent any function $\psi^{(j)} \in H^{1/2}(\partial S_j)$ ($j=1,2$) in the following form:

$$\psi^{(j)} = \begin{pmatrix} \psi^{(j)}_{L,+} \\ \psi^{(j)}_{R,-} \end{pmatrix} \oplus \begin{pmatrix} \psi^{(j)}_{L,-} \\ \psi^{(j)}_{R,+} \end{pmatrix}, \tag{3.8}$$

where indices L and R correspond to the left and right boundaries of the strip S_j.

Proposition 3.5: *If $\psi^{(j)}$ is the boundary value of a solution of the Dirac equation on S_j without AB vortex, then its components satisfy the relation*

$$\begin{pmatrix} \psi^{(j)}_{L,-} \\ \psi^{(j)}_{R,+} \end{pmatrix} = \begin{pmatrix} 0 & \hat{\omega}(\partial S_j^L|\partial S_j^R) \\ \hat{\omega}(\partial S_j^R|\partial S_j^L) & 0 \end{pmatrix}\begin{pmatrix} \psi^{(j)}_{L,+} \\ \psi^{(j)}_{R,-} \end{pmatrix}, \tag{3.9}$$

where the integral operators $\hat{\omega}(\partial S_j^L|\partial S_j^R)$ and $\hat{\omega}(\partial S_j^R|\partial S_j^L)$ are defined as follows:

$$(\hat{\omega}(\partial S_j^L|\partial S_j^R)\psi^{(j)}_{R,-})(z) = -\int_{\partial S_j^R} G^{(0)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_-\psi^{(j)}_{R,-}(z')dz' + \sigma_+\psi^{(j)}_{R,-}(z')d\bar{z}'\}, \tag{3.10}$$

$$(\hat{\omega}(\partial S_j^R|\partial S_j^L)\psi^{(j)}_{L,+})(z) = -\int_{\partial S_j^L} G^{(0)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_-\psi^{(j)}_{L,+}(z')dz' + \sigma_+\psi^{(j)}_{L,+}(z')d\bar{z}'\}. \tag{3.11}$$

- If $\psi^{(j)}$ can be continued to a solution on S_j, then, similar to (3.4), we have

$$\psi^{(j)}(z) = -\int_{\partial S_j} G^{(0)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_-\psi^{(j)}(z')dz' + \sigma_+\psi^{(j)}(z')d\bar{z}'\}$$

$$= -\int_{\partial S_j^R} G^{(0)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_-\psi^{(j)}_{R,-}(z')dz' + \sigma_+\psi^{(j)}_{R,-}(z')d\bar{z}'\} \tag{3.12}$$

$$-\int_{\partial S_j^L} G^{(0)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_-\psi^{(j)}_{L,+}(z')dz' + \sigma_+\psi^{(j)}_{L,+}(z')d\bar{z}'\}, \tag{3.13}$$

where the second equality follows from the fact that it is not possible to construct a solution on the whole disk D belonging to the domain of the free Dirac Hamiltonian $\hat{H}^{(0)}$. Passing in (3.13) to boundary values and taking the projections, we obtain the result (3.9). □

Proposition 3.6: *If* $\psi^{(j)}$ is the boundary value of a solution of the Dirac equation on $S_j \backslash \ell_j$ with only one branching point a_j, then its components satisfy the relation

$$\begin{pmatrix} \psi_{L,-}^{(j)} \\ \psi_{R,+}^{(j)} \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_{S_j}(a_j) & \hat{\beta}_{S_j}(a_j) \\ \hat{\gamma}_{S_j}(a_j) & \hat{\delta}_{S_j}(a_j) \end{pmatrix} \begin{pmatrix} \psi_{L,+}^{(j)} \\ \psi_{R,-}^{(j)} \end{pmatrix}, \tag{3.14}$$

where the integral operators $\hat{\alpha}_{S_j}(a_j)$, $\hat{\beta}_{S_j}(a_j)$ $\hat{\gamma}_{S_j}(a_j)$, and $\hat{\delta}_{S_j}(a_j)$ are defined as follows:

$$(\hat{\alpha}_{S_j}(a_j)\psi_{L,+}^{(j)})(z) = -\int_{\partial S_j^L} \dot{\Delta}^{(a_j,\nu_j)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_-\psi_{L,+}^{(j)}(z')dz' + \sigma_+\psi_{L,+}^{(j)}(z')d\bar{z}'\}, \tag{3.15}$$

$$(\hat{\beta}_{S_j}(a_j)\psi_{R,-}^{(j)})(z) = -\int_{\partial S_j^R} \dot{G}^{(a_j,\nu_j)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_-\psi_{R,-}^{(j)}(z')dz' + \sigma_+\psi_{R,-}^{(j)}(z')d\bar{z}'\}, \tag{3.16}$$

$$(\hat{\gamma}_{S_j}(a_j)\psi_{L,+}^{(j)})(z) = -\int_{\partial S_j^L} \dot{G}^{(a_j,\nu_j)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_-\psi_{L,+}^{(j)}(z')dz' + \sigma_+\psi_{L,+}^{(j)}(z')d\bar{z}'\}, \tag{3.17}$$

$$(\hat{\delta}_{S_j}(a_j)\psi_{R,-}^{(j)})(z) = -\int_{\partial S_j^R} \dot{\Delta}^{(a_j,\nu_j)}(z,z')\frac{iR}{1-|z'|^2}\{\sigma_-\psi_{R,-}^{(j)}(z')dz' + \sigma_+\psi_{R,-}^{(j)}(z')d\bar{z}'\}. \tag{3.18}$$

Here, we denote $\dot{\Delta}^{(a_j,\nu_j)}(z,z') = \dot{G}^{(a_j,\nu_j)}(z,z') - G^{(0)}(z,z')$.

- The proof is completely analogous to the previous one. □

We now choose $\psi_{L,+}^{(j)}$ and $\psi_{R,-}^{(j)}$ to be the "coordinates" in $W_j^{\text{int}}(a_j)$ $(j=1,2)$. It is not difficult to check that in these coordinates the map $F(a): W^{\text{int}}(a^0) \to W^{\text{int}}(a)$ is given by the identity operator. In order to find the representation of $P(a)^{-1}$, one should be able to decompose any $g \in W^{\text{int}}(a)$ as $g=f-h$, with $f \in W^{\text{int}}(a^0)$ and $h \in W^{\text{ext}}$. More precisely, we are interested in the relation between f and g. This calculation is very similar to the proof of the Theorem 3.1 in Ref. 4.

- First note that $h_L^{(1)}$ can be continued as a solution to the left of ∂S_1^L and, analogously, $h_R^{(2)}$ can be continued to the right of ∂S_2^R. Then one has $h_{L,+}^{(1)} = h_{R,-}^{(2)} = 0$ and, therefore,

$$g_{L,+}^{(1)} = f_{L,+}^{(1)}, \quad g_{R,-}^{(2)} = f_{R,-}^{(2)}. \tag{3.19}$$

- Next recall that the boundary values $h_R^{(1)}$ and $h_L^{(2)}$ are not independent. They are related by the formulas

$$\begin{pmatrix} h_{R,-}^{(1)} \\ h_{L,+}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & \hat{\omega}_{12} \\ \hat{\omega}_{21} & 0 \end{pmatrix} \begin{pmatrix} h_{R,+}^{(1)} \\ h_{L,-}^{(2)} \end{pmatrix},$$

$$\hat{\omega}_{12} = \hat{\omega}(\partial S_1^R | \partial S_2^L), \quad \hat{\omega}_{21} = \hat{\omega}(\partial S_2^L | \partial S_1^R),$$

which follow from Proposition 3.5. Then one finds

$$g_{R,-}^{(1)} - \hat{\omega}_{12} g_{L,-}^{(2)} = f_{R,-}^{(1)} - \hat{\omega}_{12} f_{L,-}^{(2)}, \tag{3.20}$$

$$g_{L,+}^{(2)} - \hat{\omega}_{21} g_{R,+}^{(1)} = f_{L,+}^{(2)} - \hat{\omega}_{21} f_{R,+}^{(1)}. \tag{3.21}$$

- Finally, according to Proposition 3.6, one has

$$\begin{pmatrix} f_{L,-}^{(j)} \\ f_{R,+}^{(j)} \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_{S_j}(a_j^0) & \hat{\beta}_{S_j}(a_j^0) \\ \hat{\gamma}_{S_j}(a_j^0) & \hat{\delta}_{S_j}(a_j^0) \end{pmatrix} \begin{pmatrix} f_{L,+}^{(j)} \\ f_{R,-}^{(j)} \end{pmatrix}, \tag{3.22}$$

$$\begin{pmatrix} g_{L,-}^{(j)} \\ g_{R,+}^{(j)} \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_{S_j}(a_j) & \hat{\beta}_{S_j}(a_j) \\ \hat{\gamma}_{S_j}(a_j) & \hat{\delta}_{S_j}(a_j) \end{pmatrix} \begin{pmatrix} g_{L,+}^{(j)} \\ g_{R,-}^{(j)} \end{pmatrix}. \tag{3.23}$$

If we now find $g_{R,+}^{(1)}$, $f_{R,+}^{(1)}$, $g_{L,-}^{(2)}$, and $f_{L,-}^{(2)}$ from (3.22) and (3.23) and substitute the corresponding expressions into (3.20) and (3.21), two more relations between the coordinates of f and g can be obtained. Together with (3.19), they may be written as follows:

$$\begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & -\hat{\omega}_{12}\hat{\alpha}_{S_2}(a_2) & -\hat{\omega}_{12}\hat{\beta}_{S_2}(a_2) \\ -\hat{\omega}_{21}\hat{\gamma}_{S_1}(a_1) & -\hat{\omega}_{21}\hat{\delta}_{S_1}(a_1) & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} g_{L,+}^{(1)} \\ g_{R,-}^{(1)} \\ g_{L,+}^{(2)} \\ g_{R,-}^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & -\hat{\omega}_{12}\hat{\alpha}_{S_2}(a_2^0) & -\hat{\omega}_{12}\hat{\beta}_{S_2}(a_2^0) \\ -\hat{\omega}_{21}\hat{\gamma}_{S_1}(a_1^0) & -\hat{\omega}_{21}\hat{\delta}_{S_1}(a_1^0) & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} f_{L,+}^{(1)} \\ f_{R,-}^{(1)} \\ f_{L,+}^{(2)} \\ f_{R,-}^{(2)} \end{pmatrix} \tag{3.24}$$

or, in a more compact form,

$$(\mathbf{1} + M(a))g = (\mathbf{1} + M(a^0))f. \tag{3.25}$$

Therefore, the τ-function is given by

$$\tau(a,a^0) = \det_{W^{\text{int}}(a^0)}(\mathbf{1} + M(a^0))^{-1}(\mathbf{1} + M(a)). \tag{3.26}$$

Remark: One-point Green's function $\dot{G}^{(a_j,\nu_j)}(z,z')$ is related to the one-vortex Green's function $G(z,z')$, calculated in Sec. II, by a simple unitary (+ singular gauge) transformation. For example, one has

$$\dot{\Delta}^{(a_j=0,\nu_j=\nu)}(z,z') = e^{i\nu(\varphi-\varphi')}\Delta(z,z'). \tag{3.27}$$

Thus the action of all operators in (3.24) is known. However, in order to obtain from (3.26) an explicit formula for the τ-function, one still needs to introduce some coordinates in the infinite-dimensional spaces $(H^{1/2}(\partial S_{1,2}^{L,R}))_{\pm}$. It turns out, though (see Theorem 6.3 in Ref. 14 and Sec. III D 3 of the present paper) that the logarithmic derivative of the τ-function (3.6) does not depend on localization (i.e., on the choice of S) and on a^0. Moreover, since $\tau(a^0,a^0)=1$, one should have

$$\tau(a,a^0) = \frac{\tau(a)}{\tau(a^0)}.$$

Therefore, one may fix the coordinates in $(H^{1/2}(\partial S_{1,2}^{L,R}))_{\pm}$ and calculate the τ-function for a convenient choice of S. These problems are addressed in Secs. III B and III E.

B. Dirac equation on the Poincaré strip

Coordinate change $z=\tanh\xi$ maps the Poincaré disk onto the strip $\mathcal{U}=\{\xi:|\xi_y|<\pi/4\}$ in the complex ξ-plane. In these coordinates, the Poincaré metric is given by

$$ds^2 = g_{\xi\bar{\xi}}d\xi d\bar{\xi} = R^2\frac{d\xi d\bar{\xi}}{\cosh^2(\xi-\bar{\xi})},$$

and the vector potential of the uniform magnetic field B can be chosen in the form

$$\mathcal{A}^{(B)}_{\text{strip}} = -2b \tan 2\xi_y d\xi_x. \tag{3.28}$$

The corresponding Dirac Hamiltonian $\hat{H}^{(0)}_{\text{strip}}$ is explicitly given by the formula (2.3) with

$$K = \frac{1}{R}\{2 \cos 2\xi_y \partial_{\xi} - i(1+2b)\sin 2\xi_y\}, \tag{3.29}$$

$$K^* = -\frac{1}{R}\{2 \cos 2\xi_y \partial_{\bar{\xi}} + i(1-2b)\sin 2\xi_y\}. \tag{3.30}$$

It is related to the free Dirac Hamiltonian on the disk by a unitary transformation

$$\hat{H}^{(0)}_{\text{strip}}(\xi) = U_{\text{DS}}(\xi)\hat{H}^{(0)}_{\text{disk}}(z \mapsto \tanh \xi)U^{\dagger}_{\text{DS}}(\xi), \tag{3.31}$$

$$U_{\text{DS}}(\xi) = \begin{pmatrix} \left(\dfrac{\cosh \xi}{\cosh \bar{\xi}}\right)^{-(1-2b)/2} & 0 \\ 0 & \left(\dfrac{\cosh \xi}{\cosh \bar{\xi}}\right)^{(1+2b)/2} \end{pmatrix}. \tag{3.32}$$

The main advantage of the gauge (3.28) is that $\hat{H}^{(0)}_{\text{strip}}$ commutes with the ξ_x-momentum operator $\hat{P}_x = -i\partial_{\xi_x}$. The eigenspace of $\hat{P}_x$, characterized by the momentum $p \in \mathbb{R}$, is composed of the spinors of the form $g(p,\xi_y)e^{ip\xi_x}$. Being restricted to this eigenspace, the Hamiltonian $\hat{H}^{(0)}_{\text{strip}}$ acts as follows:

$$g(p,\xi_y) \mapsto \hat{H}_p g(p,\xi_y), \quad \hat{H}_p = R^{-1}\begin{pmatrix} mR & K_p \\ K^*_p & -mR \end{pmatrix}, \tag{3.33}$$

where the operators K_p and K^*_p are given by

$$K_p = -i[\cos 2\xi_y(\partial_{\xi_y} - p) + (1+2b)\sin 2\xi_y], \tag{3.34}$$

$$K^*_p = -i[\cos 2\xi_y(\partial_{\xi_y} + p) + (1-2b)\sin 2\xi_y]. \tag{3.35}$$

Let us consider the partial Dirac equation,

$$(\hat{H}_p - E)g(p,\xi_y) = 0, \tag{3.36}$$

where it is assumed that E is real and $|E| < m$. Two linearly independent solutions of (3.36) can be chosen in the following way:

$$\Phi^{(\pm)}(p,\xi_y) = (2 \cos 2\xi_y)^{(1+2\mu)/2}\sqrt{\chi(p)} \times \begin{pmatrix} C_+^{-1} e^{\pm i(2\mu+2b\pm ip)(\xi_y \mp \pi/4)} {}_2\tilde{F}_1\left(\mu+b, \mu+\frac{1}{2} \pm \frac{ip}{2}, 1+2\mu, 1+e^{\pm 4i\xi_y}\right) \\ \pm iC_+ e^{\pm i(2\mu-2b\mp ip)(\xi_y \mp \pi/4)} {}_2\tilde{F}_1\left(\mu-b, \mu+\frac{1}{2} \mp \frac{ip}{2}, 1+2\mu, 1+e^{\pm 4i\xi_y}\right) \end{pmatrix}, \tag{3.37}$$

with

$$\chi(p) = \frac{\Gamma(\mu - b + 1)\Gamma(\mu + b + 1)\Gamma\left(\mu + \frac{1}{2} - \frac{ip}{2}\right)\Gamma\left(\mu + \frac{1}{2} + \frac{ip}{2}\right)}{4\pi(\Gamma(1 + 2\mu))^2}.$$

The "$\sim$" in ${}_2\widetilde{F}_1$ indicates that the hypergeometric function is defined not on its principal branch but on the cut plane $\mathbb{C}\backslash(-\infty,1]$ with $\lim_{z\to 0,\mathrm{Im}\, z>0}\, {}_2\widetilde{F}_1(a,b,c,z)=1$, as in the formula (3.5) of Ref. 17. Notice that the first solution, $\Phi^{(+)}(p,\xi_y)$, satisfies the condition of square integrability on the upper edge of the strip $\mathcal{U}$ ($\xi_y=\pi/4$) and the second one, $\Phi^{(-)}(p,\xi_y)$, does so on the lower edge ($\xi_y=-\pi/4$):

$$\Phi^{(\pm)}\left(p,\xi_y \to \pm\frac{\pi}{4}\right) = (2\cos 2\xi_y)^{(1+2\mu)/2}\sqrt{\chi(p)}\begin{pmatrix} C_+^{-1} \\ \pm iC_+ \end{pmatrix} + O((\cos 2\xi_y)^{(3+2\mu)/2}). \qquad (3.38)$$

These two solutions verify simple symmetry relations,

$$\Phi^{(+)}(p,\xi_y) = \sigma_z\Phi^{(-)}(-p,-\xi_y), \quad \Phi^{(\pm)}(p,\xi_y) = \sigma_z\overline{\Phi^{(\pm)}(p,\xi_y)}. \qquad (3.39)$$

It is also worthwhile to give a formula for the determinant of the fundamental matrix built from $\Phi^{(+)}(p,\xi_y)$ and $\Phi^{(-)}(p,\xi_y)$. Using transformation formulas for hypergeometric functions, one obtains

$$\det(\Phi^{(+)}(p,\xi_y),\Phi^{(-)}(p,\xi_y)) = -i\cos 2\xi_y. \qquad (3.40)$$

Green's function $G_{E,p}(\xi_y,\xi_y')$ of the partial Hamiltonian $\hat{H}_p$ satisfies the equation

$$(\hat{H}_p(\xi) - E)G_{E,p}(\xi_y,\xi_y') = \frac{\cos^2 2\xi_y}{R^2}\delta(\xi_y - \xi_y')\mathbf{1}_2. \qquad (3.41)$$

Analogously to Sec. II C, consider the ansatz

$$G_{E,p}(\xi_y,\xi_y') = \begin{cases} C_{E,p}\Phi^{(-)}(p,\xi_y)\otimes(\Phi^{(+)}(p,\xi_y'))^\dagger & \text{for} \quad -\frac{\pi}{4} < \xi_y < \xi_y' < \frac{\pi}{4} \\ C_{E,p}\Phi^{(+)}(p,\xi_y)\otimes(\Phi^{(-)}(p,\xi_y'))^\dagger & \text{for} \quad -\frac{\pi}{4} < \xi_y' < \xi_y < \frac{\pi}{4}. \end{cases} \qquad (3.42)$$

It solves (3.41) for $\xi_y \neq \xi_y'$ and satisfies the appropriate boundary conditions as $\xi_y,\xi_y' \to \pm\pi/4$. The required singular behavior at $\xi_y=\xi_y'$ is equivalent to the condition

$$G_{E,p}(\xi_y + 0,\xi_y) - G_{E,p}(\xi_y - 0,\xi_y) = i\frac{\cos 2\xi_y}{R}\sigma_x. \qquad (3.43)$$

Substituting (3.42) into the last relation and using symmetry properties (3.39), one may show that (3.43) holds true if we choose $C_{E,p}=R^{-1}$.

C. Fredholm determinant representations

1. Boundary projections revisited

Let us fix a line $\mathcal{L}_{\xi_y^{(0)}}=\{\xi\in\mathcal{U}\,|\,\xi_y=\xi_y^{(0)}\}$ and consider a $\mathbb{C}^2$-valued function $g_{\xi_y^{(0)}}(\xi_x)\in H^{1/2}(\mathcal{L}_{\xi_y^{(0)}})$, represented by its Fourier decomposition

$$g_{\xi_y^{(0)}}(\xi_x) = \int_{-\infty}^{\infty} dp\, g(p,\xi_y^{(0)})e^{ip\xi_x}. \qquad (3.44)$$

Next, we introduce two operators, $Q_\pm(\xi_y^{(0)})$, whose action on Fourier transform is given by a matrix multiplication

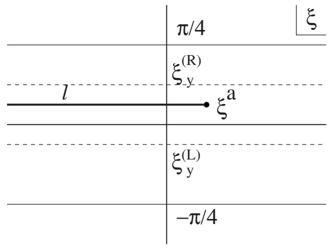

FIG. 3. Poincaré strip $|\mathrm{Im}\,\xi| < \pi/4$.

$$Q_\pm(\xi_y^{(0)})g(p,\xi_y^{(0)}) = Q_\pm(p,\xi_y^{(0)})g(p,\xi_y^{(0)}),$$

with

$$Q_\pm(p,\xi_y^{(0)}) = \mp \frac{i}{\cos 2\xi_y^{(0)}}\Phi^{(\pm)}(p,\xi_y^{(0)}) \otimes (\Phi^{(\mp)}(p,\xi_y^{(0)}))^\dagger \sigma_x. \tag{3.45}$$

Symmetry properties (3.39) and the formula (3.40) imply the following result.

Proposition 3.7: *The operators $Q_\pm(p,\xi_y^{(0)})$ satisfy the relations*

$$(Q_\pm(p,\xi_y^{(0)}))^2 = Q_\pm(p,\xi_y^{(0)}), \quad Q_+(p,\xi_y^{(0)}) + Q_-(p,\xi_y^{(0)}) = \mathbf{1}_2. \tag{3.46}$$

Thus one has a decomposition $H^{1/2}(\mathcal{L}_{\xi_y^{(0)}}) = H_+(\xi_y^{(0)}) \oplus H_-(\xi_y^{(0)})$, where $H_\pm(\xi_y^{(0)}) = Q_\pm(\xi_y^{(0)})H^{1/2}(\mathcal{L}_{\xi_y^{(0)}})$. We remark at once that $H_\pm(\xi_y^{(0)})$ will play the role of the subspaces $(H^{1/2}(\partial S_{1,2}^{L,R}))_\pm$ from Sec. III A since they are composed of the boundary values of functions ψ satisfying the Dirac equation $(\hat{H}_{\mathrm{strip}}^{(0)} - E)\psi = 0$ and the condition of square integrability in the region $\xi_y > \xi_y^{(0)}$ (in the case of H_+) or $\xi_y < \xi_y^{(0)}$ (for H_-).

Using the formulas (3.45) and (3.46), one may write Fourier transform of $g_{\xi_y^{(0)}}(\xi_x)$ as follows:

$$g(p,\xi_y^{(0)}) = \tilde{g}_+(p,\xi_y^{(0)})\Phi^{(+)}(p,\xi_y^{(0)}) + \tilde{g}_-(p,\xi_y^{(0)})\Phi^{(-)}(p,\xi_y^{(0)}),$$

where

$$\tilde{g}_\pm(p,\xi_y^{(0)}) = \mp \frac{i}{\cos 2\xi_y^{(0)}}(\Phi^{(\mp)}(p,\xi_y^{(0)}))^\dagger \sigma_x g(p,\xi_y^{(0)}). \tag{3.47}$$

It is convenient to think of $\tilde{g}_\pm(p,\xi_y^{(0)})$ as coordinates of $g_{\xi_y^{(0)}}$ in $H_\pm(\xi_y^{(0)})$. It should be noted that $\tilde{g}_\pm(p,\xi_y^{(0)})$ are ordinary functions, in contrast to (3.8), where a similar notation was used for 2-columns. In such coordinates, Propositions 3.5 and 3.6 have the following form.

Proposition 3.8: *Let us consider a strip $\mathcal{S} = \{\xi \in \mathcal{U} | \xi_y^{(L)} < \xi_y < \xi_y^{(R)}\}$. Suppose that a $\mathbb{C}^2$-valued function $\psi \in H^{1/2}(\partial\mathcal{S})$ can be continued to $\mathcal{S}$ as a solution of the Dirac equation $(\hat{H}_{\mathrm{strip}}^{(0)} - E)\psi = 0$. Then, one has*

$$\begin{pmatrix} \tilde{\psi}_{L,-}(p) \\ \tilde{\psi}_{R,+}(p) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}_{L,+}(p) \\ \tilde{\psi}_{R,-}(p) \end{pmatrix}. \tag{3.48}$$

Proposition 3.9: *Let us now assume that the strip $\mathcal{S}$ contains one branching point ξ^a and introduce a horizontal branch cut $\ell \in \mathcal{S}$, as shown in Fig. 3. Suppose that $\psi \in H^{1/2}(\partial\mathcal{S})$ is the boundary value of a multivalued solution of the Dirac equation on $\mathcal{S}\backslash\ell$, characterized by the monodromy $e^{2\pi i\nu}$ at the point ξ^a. Then,*

$$\begin{pmatrix} \tilde{\psi}_{L,-}(p) \\ \tilde{\psi}_{R,+}(p) \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_{\mathcal{S}}(\xi^a) & \hat{\beta}_{\mathcal{S}}(\xi^a) \\ \hat{\gamma}_{\mathcal{S}}(\xi^a) & \hat{\delta}_{\mathcal{S}}(\xi^a) \end{pmatrix} \begin{pmatrix} \tilde{\psi}_{L,+}(p) \\ \tilde{\psi}_{R,-}(p) \end{pmatrix},$$

where

$$(\hat{\alpha}_{\mathcal{S}}(\xi^a)\tilde{\psi}_{L,+})(p) = R\int_{-\infty}^{\infty} \dot{\Delta}_{-}^{(\xi^a,\nu)}(p,q)\tilde{\psi}_{L,+}(q)dq, \tag{3.49}$$

$$(\hat{\beta}_{\mathcal{S}}(\xi^a)\tilde{\psi}_{R,-})(p) = R\int_{-\infty}^{\infty} \dot{G}_{+}^{(\xi^a,\nu)}(p,q)\tilde{\psi}_{R,-}(q)dq, \tag{3.50}$$

$$(\hat{\gamma}_{\mathcal{S}}(\xi^a)\tilde{\psi}_{L,+})(p) = R\int_{-\infty}^{\infty} \dot{G}_{-}^{(\xi^a,\nu)}(p,q)\tilde{\psi}_{L,+}(q)dq, \tag{3.51}$$

$$(\hat{\delta}_{\mathcal{S}}(\xi^a)\tilde{\psi}_{R,-})(p) = R\int_{-\infty}^{\infty} \dot{\Delta}_{+}^{(\xi^a,\nu)}(p,q)\tilde{\psi}_{R,-}(q)dq. \tag{3.52}$$

and

$$\dot{\Delta}_{\pm}^{(\xi^a,\nu)}(p,q) = \frac{1}{2\pi\cos 2\xi_y \cos 2\xi_y'} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi_x d\xi_x' \;\; e^{-ip\xi_x+iq\xi_x'}(\Phi^{(\mp)}(p,\xi_y))^{\dagger}\sigma_x \dot{\Delta}_{\text{strip}}^{(\xi^a,\nu)}(\xi,\xi')\big|_{\xi_y,\xi_y'\gtrless\xi_y^a}\sigma_x\Phi^{(\mp)}(q,\xi_y'), \tag{3.53}$$

$$\dot{G}_{\pm}^{(\xi^a,\nu)}(p,q) = -\frac{1}{2\pi\cos 2\xi_y \cos 2\xi_y'} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi_x d\xi_x' e^{-ip\xi_x+iq\xi_x'}(\Phi^{(\pm)}(p,\xi_y))^{\dagger}\sigma_x \dot{G}_{\text{strip}}^{(\xi^a,\nu)}(\xi,\xi')\big|_{\xi_y\lessgtr\xi_y^a,\xi_y'\gtrless\xi_y^a}\sigma_x\Phi^{(\mp)}(q,\xi_y'). \tag{3.54}$$

Here, $\dot{G}_{\text{strip}}^{(\xi^a,\nu)}(\xi,\xi')$ denotes the Green's function of the Dirac Hamiltonian on the strip with one branching point ξ^a, and $\dot{\Delta}_{\text{strip}}^{(\xi^a,\nu)}(\xi,\xi') = \dot{G}_{\text{strip}}^{(\xi^a,\nu)}(\xi,\xi') - G_{\text{strip}}^{(0)}(\xi,\xi')$.

- We illustrate the idea by deriving (3.49). On the strip, the formula (3.15) is replaced by

$$(\hat{\alpha}_{\mathcal{S}}(\xi^a)\psi_{L,+})(\xi) = -\frac{iR}{\cos 2\xi_y^{(L)}}\int_{-\infty}^{\infty} \dot{\Delta}_{\text{strip}}^{(\xi^a,\nu)}(\xi,\xi')\sigma_x\psi_{L,+}(\xi')d\xi_x',$$

with $\xi_y = \xi_y' = \xi_y^{(L)}$. Then after Fourier transform, one obtains

$$(\hat{\alpha}_{\mathcal{S}}(\xi^a)\tilde{\psi}_{L,+})(p)\cdot\Phi^{(-)}(p,\xi_y^{(L)}) = -\frac{iR}{\cos 2\xi_y^{(L)}}\int_{-\infty}^{\infty} dq\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi_x d\xi_x' \times \dot{\Delta}_{\text{strip}}^{(\xi^a,\nu)}(\xi,\xi')e^{-ip\xi_x+iq\xi_x'}\right\}\sigma_x\Phi^{(+)}(q,\xi_y^{(L)})\tilde{\psi}_{L,+}(q). \tag{3.55}$$

The columns of the expression in curly brackets satisfy partial Dirac equation $(\hat{H}_p(\xi)-E)\psi = 0$ and, in addition, for $\xi_y < \xi_y^a$ they are square integrable as $\xi_y \to -\pi/4$. Hence, they are both proportional to $\Phi^{(-)}(p,\xi_y)$. Similarly, the rows of this expression are proportional to $(\Phi^{(-)}(q,\xi_y'))^{\dagger}$. Therefore, for $\xi_y, \xi_y' < \xi_y^a$, one has

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi_x d\xi_x' \dot{\Delta}_{\text{strip}}^{(\xi^a,\nu)}(\xi,\xi')e^{-ip\xi_x+iq\xi_x'} = \dot{\Delta}_{-}^{(\xi^a,\nu)}(p,q)\Phi^{(-)}(p,\xi_y)\otimes(\Phi^{(-)}(q,\xi_y'))^{\dagger}, \tag{3.56}$$

and it is a simple matter to verify that the coefficient $\dot{\Delta}_{-}^{(\xi^a,\nu)}(p,q)$ is indeed given by (3.53). Substituting (3.56) into (3.55), we obtain (3.49). □

Finally, suppose that the Poincaré strip $\mathcal{U}$ contains two branching points, ξ^{a_1} and ξ^{a_2}, such that $\tanh \xi^{a_j}=a_j$ $(j=1,2)$ and $\xi_y^{a_1}<\xi_y^{a_2}$. Using (3.31) and going through the definitions of Sec. III A, one finds that the two-point tau function $\tau(a)$ can be written as a Fredholm determinant,

$$\tau(a) = \det(\mathbf{1} - \hat{\alpha}(\xi^{a_2})\hat{\delta}(\xi^{a_1})), \tag{3.57}$$

The kernels (3.49) and (3.52) of the integral operators $\hat{\alpha}(\xi^{a_2})$ and $\hat{\delta}(\xi^{a_1})$ do not depend on the choice of the strips $\mathcal{S}_1$ and $\mathcal{S}_2$; hence, the corresponding indices will be omitted from now on.

Remark: It turns out that the tau function depends only on the geodesic distance $d(a_1,a_2)$ between the points a_1 and a_2 (see Appendix). Therefore, to simplify (3.57), we may choose $\xi^{a_1}=0$, $\xi^{a_2}=l_s+i0$, and $l_s=[d(a_1,a_2)]/R\in\mathbb{R}^+$. In addition, the invariance of $\hat{H}_{\text{strip}}^{(0)}$ with respect to ξ_x-translations implies that

$$\dot{\Delta}_{\pm}^{(\xi_x^a+l_s+i\xi_y^a,\nu)}(p,q) = e^{i(p-q)l_s}\dot{\Delta}_{\pm}^{(\xi_x^a+i\xi_y^a,\nu)}(p,q).$$

Thus it remains to compute only the quantities $\dot{\Delta}_{\pm}^{(0,\nu)}(p,q)$, henceforth referred to as "form factors." This problem is solved in Sec. III C 2.

2. Asymptotics of $\dot{\Delta}(\xi,\xi')$

Let us first compute $\dot{\Delta}_{-}^{(0,\nu)}(p,q)$. The idea is to use instead of (3.53) the equivalent formula (3.56). Since it is valid for all $\xi_y,\xi_y'<0$, one may obtain $\dot{\Delta}_{-}^{(0,\nu)}(p,q)$ by exploring the asymptotics of both sides of this relation as $\xi_y,\xi_y'\to-\pi/4$. The asymptotics of the right hand side of (3.56) is determined by the formula (3.38), which gives

$$\text{leading behavior of the RHS of Eq. 3.56 as } \xi_y,\xi_y'\to-\frac{\pi}{4}\simeq\dot{\Delta}_{-}^{(0,\nu)}(p,q)\sqrt{\chi(p)\chi(q)} \times(4\cos 2\xi_y\cos 2\xi_y')^{(1+2\mu)/2}\begin{pmatrix} C_+^{-2} & i \\ -i & C_+^2\end{pmatrix}. \tag{3.58}$$

To find the asymptotics of the left hand side, we will use the results of Sec. II. Set for definiteness $\Theta=-\pi/2$. The formula (2.47) then implies that to a leading order

$$\Delta(z,z')|_{r,r'\to1,|\varphi-\varphi'|<\pi}\simeq\frac{\sin\pi\nu}{2\pi^2R}\frac{\Gamma(\mu-b+1)\Gamma(\mu+b+1)}{\Gamma(1+2\mu)}\begin{pmatrix} C_+^{-2} & e^{-i\varphi'} \\ e^{i\varphi} & C_+^2e^{i(\varphi-\varphi')}\end{pmatrix} \times[(1-r^2)(1-r'^2)]^{(1+2\mu)/2}\int_{-\infty}^{\infty}d\theta\frac{e^{(1+\nu)\theta+i(\varphi-\varphi')}}{e^{\theta+i(\varphi-\varphi')}+1}\frac{e^{(1-2b)\theta/2}}{(2+2\cosh\theta)^{(1+2\mu)/2}}. \tag{3.59}$$

In order to obtain the asymptotics of $\dot{\Delta}_{\text{strip}}^{(0,\nu)}(\xi,\xi')$ as $\xi_y,\xi_y'\to-\pi/4$ from (3.59), one should take into account the following comments.

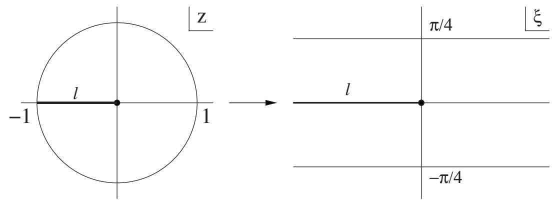

FIG. 4. Mapping of the Poincaré disk onto the strip.

- Dirac Hamiltonian on the disk with one AB vortex is related to the free Hamiltonian on the disk with one branching point by a singular gauge transformation [see (3.27)]. The choice of the branch cut represented in Fig. 4 is equivalent to saying that φ and φ' vary between $-\pi$ and π.
- Free Hamiltonians on the disk and on the strip are related by the unitary transformation (3.32). Also notice that

$$U_{\mathrm{DS}}(\xi \mapsto \operatorname{arctanh} z)|_{\xi_y \to -\pi/4} \simeq \begin{pmatrix} e^{i[(1-2b)/2](\varphi+\pi/2)} & 0 \\ 0 & e^{-i[(1+2b)/2](\varphi+\pi/2)} \end{pmatrix}, \quad \varphi \in (-\pi, 0).$$

- The integrals over ξ_x and ξ'_x in (3.59) can be written as integrals over φ and φ' using the formulas

$$e^{2\xi_x}|_{\xi_y \to -\pi/4} \simeq \operatorname{ctg}\left(-\frac{\varphi}{2}\right), \quad 2\cos 2\xi_y|_{\xi_y \to -\pi/4} \simeq \frac{1-r^2}{\sin(-\varphi)}, \quad d\xi_x|_{\xi_y \to -\pi/4} \simeq \frac{d\varphi}{2\sin(-\varphi)},$$

where, as above, $\varphi \in (-\pi, 0)$.

Once the leading behavior of $\dot{\Delta}^{(0,\nu)}_{\mathrm{strip}}(\xi,\xi')$ is found, one should substitute it into (3.56). Then, comparing the left hand side of (3.56) with (3.58), we obtain a triple integral representation for the form factors

$$R\dot{\Delta}^{(0,\nu)}_{-}(p,q) = \sqrt{\rho(p)\rho(q)}\mathcal{F}_\nu(p,q), \tag{3.60}$$

where

$$\rho(p) = \frac{2^{2\mu}\Gamma(1+2\mu)}{\Gamma\left(\mu+\frac{1}{2}+\frac{ip}{2}\right)\Gamma\left(\mu+\frac{1}{2}-\frac{ip}{2}\right)}, \tag{3.61}$$

$$\mathcal{F}_\nu(p,q) = \frac{\sin \pi\nu}{2\pi^2}\int_{-\infty}^{\infty} d\theta \int_0^{\pi/2} dx \int_0^{\pi/2} dy \frac{e^{\{1+\nu+[(1-2b)/2]\}(\theta-2i(x-y))}}{(e^{\theta-2i(x-y)}+1)(2+2\cosh\theta)^{(1+2\mu)/2}}$$
$$\times (\sin x)^{\mu+(ip/2)-1/2}(\cos x)^{\mu-(ip/2)-1/2}(\sin y)^{\mu-(iq/2)-1/2}(\cos y)^{\mu+(iq/2)-1/2}. \tag{3.62}$$

One may deduce from (3.60)–(3.62) the following symmetry properties:

$$\dot{\Delta}^{(0,\nu)}_{-}(p,q) = \overline{\dot{\Delta}^{(0,\nu)}_{-}(p,q)} = \dot{\Delta}^{(0,\nu)}_{-}(q,p). \tag{3.63}$$

Another useful representation for $\dot{\Delta}^{(0,\nu)}_{-}(p,q)$ can be found by decomposing the integral over θ in (3.62) into two pieces, $\int_{-\infty}^{0}$ and $\int_0^\infty$, and replacing $(e^{\theta-2i(x-y)}+1)^{-1}$ with the appropriate geometric series in each piece. Then the integrals over x, y, and θ can be computed in terms of hypergeometric functions and we obtain

$$R\dot{\Delta}_-^{(0,\nu)}(p,q) = 2^{2\mu-1}\frac{\sin \pi\nu}{\pi^2}\frac{\left[\Gamma\left(\mu+\frac{1}{2}+\frac{ip}{2}\right)\Gamma\left(\mu+\frac{1}{2}-\frac{ip}{2}\right)\Gamma\left(\mu+\frac{1}{2}+\frac{iq}{2}\right)\Gamma\left(\mu+\frac{1}{2}-\frac{iq}{2}\right)\right]^{1/2}}{\Gamma(1+2\mu)}$$
$$\times\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}e^{-\pi(p+q)/4}}{\mu+1+\nu-b+n}{}_2F_1\left(\mu+1+\nu-b+n,\mu+\frac{1}{2}+\frac{ip}{2},1+2\mu,2-i0\right)\right.$$
$$\times{}_2F_1\left(\mu+1+\nu-b+n,\mu+\frac{1}{2}-\frac{iq}{2},1+2\mu,2+i0\right)$$
$$\times{}_2F_1(\mu+1+\nu-b+n,1+2\mu,\mu+2+\nu-b+n,-1)$$
$$+\sum_{n=0}^{\infty}\frac{(-1)^{n}e^{\pi(p+q)/4}}{\mu-\nu+b+n}{}_2F_1\left(\mu-\nu+b+n,\mu+\frac{1}{2}+\frac{ip}{2},1+2\mu,2+i0\right)$$
$$\times{}_2F_1\left(\mu-\nu+b+n,\mu+\frac{1}{2}-\frac{iq}{2},1+2\mu,2-i0\right){}_2F_1(\mu-\nu+b+n,1+2\mu,\mu-\nu+b$$
$$\left.+n+1,-1)\right\}. \tag{3.64}$$

It is worthwhile to note that both series in (3.64) rapidly converge since for large n their nth terms tend to 0 as at least $n^{-2-2\mu}$.

Remark: We have obtained the representations (3.60)–(3.62) and (3.64) assuming that $p,q \in \mathbb{R}$. However, they also give an analytic continuation of $\dot{\Delta}_-^{(0,\nu)}(p,q)$ to the strips $|\mathrm{Im}\, p|,|\mathrm{Im}\, q| < 1+2\mu$. The extension to the whole complex p- and q-plane may also be constructed: in particular, it can be shown that $\mathcal{F}_\nu(p,q)$ has only simple poles at $p=\pm i(1+2\mu+2n_1)$, $q=\pm i(1+2\mu+2n_2)$, where $n_{1,2}=0,1,2,\ldots$. This singularity structure will be used later in the study of the long-distance behavior of the τ-function.

The calculation of $\dot{\Delta}_+^{(0,\nu)}(p,q)$ is quite similar, the result being simply $\dot{\Delta}_+^{(0,\nu)}(p,q)=\dot{\Delta}_-^{(0,\nu)}(-p,-q)$. Finally, for another distinguished value of the SAE parameter, $\Theta=\pi/2$, the representation (2.49) for the corresponding Green's function implies that

$$R\dot{\Delta}_\pm^{(0,\nu)}(p,q) = \sqrt{\rho(p)\rho(q)}\mathcal{F}_{\nu-1}(\mp p, \mp q). \tag{3.65}$$

Thus we can now compute the τ-function of the Dirac Hamiltonian for all four types of boundary conditions.

Example: Let us choose $\xi^{a_1}=0$, $\xi^{a_2}=l_s+i0$, as in the remark at the end of Sec. III C 1. Recall that $l_s=\operatorname{arctanh}\sqrt{s}$ denotes rescaled geodesic distance between the points $a_1=0$ and $a_2=s$. We also assume that the upper component of the functions from the domain of the Dirac Hamiltonian is regular as $\xi\to\xi^{a_1},\xi^{a_2}$. Then the τ-function (3.57) can be written as follows:

$$\tau(s) = \det(\mathbf{1}-L_{\nu_2,s}L'_{\nu_1,s}), \tag{3.66}$$

where the kernels of the integral operators $L_{\nu,s}$ and $L'_{\nu,s}$ are

$$L_{\nu,s}(p,q) = e^{i(p-q)l_s/2}\sqrt{\rho(p)\rho(q)}\mathcal{F}_\nu(p,q), \tag{3.67}$$

$$L'_{\nu,s}(p,q) = L_{\nu,s}(-p,-q), \tag{3.68}$$

and $p,q\in\mathbb{R}$. The functions $\rho(p)$ and $\mathcal{F}_\nu(p,q)$ are given by (3.61) and (3.62).

D. Relation to Painlevé VI

Let us briefly describe the relation of the present paper to the PBT work.[14] Recall that the Hamiltonian $\hat{H}^{(0)}$ of a Dirac particle in the absence of the AB fluxes is given by the formula (2.3) with

$$K = R^{-1}[2(1-|z|^2)\partial_z + (1+2b)\bar{z}],$$

$$K^* = -R^{-1}[2(1-|z|^2)\partial_{\bar{z}} + (1-2b)z].$$

Consider the operator

$$\hat{A} = U(\hat{H}^{(0)} - E)U\sigma_z, \quad U = \mathrm{diag}\left(\left(\frac{m+E}{m-E}\right)^{1/4}, \left(\frac{m+E}{m-E}\right)^{-1/4}\right).$$

It is straightforward to check that $\hat{A}$ coincides with the operator $m-D_k$ studied by PBT [see, e.g., the formulas (1.14)–(1.16) in Ref. 14] if we identify

$$m_{\mathrm{PBT}} = \sqrt{m^2-E^2}, \quad k_{\mathrm{PBT}} = -b.$$

In the presence of branch points, one should only replace $\hat{H}^{(0)}$ in the definition of $\hat{A}$ by the operator $\hat{H}^{(a,\nu)}$, introduced in the beginning of this section (recall that $\hat{H}^{(a,\nu)}$ is obtained from the Dirac Hamiltonian with AB field by a singular gauge transformation). Thus there is a unique correspondence between the multivalued solutions of the Dirac equation considered in Ref. 14 and the solutions of $(\hat{H}^{(a,\nu)}-E)\psi=0$. Using this correspondence, we now reformulate the key steps of the PBT analysis in the context of the present work.

1. Symmetries and elementary solutions

The Hamiltonian $\hat{H}^{(0)}$ transforms covariantly under the action of the isometry group of the Poincaré disk. In particular, if $\psi(z)$ satisfies the equation $(\hat{H}^{(0)}-E)\psi=0$, then for any $g=\left(\begin{smallmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{smallmatrix}\right) \in SU(1,1)$ the function $\psi_g(z)$ defined by

$$\psi_g(z) = \begin{pmatrix} v(g,z)^{(1-2b)/2} & 0 \\ 0 & v(g,z)^{-[(1+2b)/2]} \end{pmatrix} \psi(z_{g^{-1}}(z)), \quad v(g,z) = \frac{\bar{\alpha}-\beta\bar{z}}{\alpha-\bar{\beta}z}, \tag{3.69}$$

is a solution of the same equation. A basis in the Lie algebra of the corresponding complexified infinitesimal symmetries can be chosen in the following way:

$$M_+ = -z^2\partial_z + \partial_{\bar{z}} + bz - \frac{z}{2}\sigma_z,$$

$$M_- = \partial_z - \bar{z}^2\partial_{\bar{z}} - b\bar{z} + \frac{\bar{z}}{2}\sigma_z,$$

$$M_3 = z\partial_z - \bar{z}\partial_{\bar{z}} - b + \tfrac{1}{2}\sigma_z = \hat{L} - b.$$

These generators satisfy $sl(2)$ commutation relations

$$[M_3, M_\pm] = \pm M_\pm, \quad [M_+, M_-] = 2M_3$$

and, therefore, one may introduce two families of solutions $\{w_l\}$, $\{w_l^*\}$ of the Dirac equation on D with one branch point at $z=0$, on which the symmetries act as follows:

$$M_3 w_l = (l-b) w_l, \quad M_3 w_l^* = -(l+b) w_l^*,$$

$$M_+ w_l = m_+(l,b) w_{l+1}, \quad M_+ w_l^* = w_{l-1}^*,$$

$$M_- w_l = w_{l-1}, \quad M_- w_l^* = m_-(l,b) w_{l+1}^*. \tag{3.70}$$

These relations fix $\{w_l\}$ and $\{w_l^*\}$ up to overall normalization. Since $\{w_l\}$ and $\{w_l^*\}$ diagonalize both the Hamiltonian and angular momentum, they are related to the radial wave functions introduced in Sec. II. It is convenient to choose

$$w_l(z) = \frac{e^{-i\pi l}}{2\pi} \frac{\Gamma(\mu - b + 1)}{\Gamma(\mu - b + l + 1/2)} \mathrm{w}_{l-1/2}^{(\mathrm{II},+)}(z),$$

$$w_l^*(z) = \frac{e^{i\pi l}}{2\pi} \frac{\Gamma(\mu + b + 1)}{\Gamma(\mu + b + l + 1/2)} \mathrm{w}_{-l-1/2}^{(\mathrm{II},-)}(z),$$

and it then follows that $m_\pm(l,b) = \mu^2 - (b \mp l \mp 1/2)^2$. Using (3.69), we also introduce the elementary solutions with one branch point at $z=a$,

$$w_l(z,a) = V(T[a],z) w_l(T[-a]z), \quad w_l^*(z,a) = V(T[a],z) w_l^*(T[-a]z),$$

where

$$T[-a]z = \frac{z-a}{1-\bar{a}z}, \quad V(T[a],z) = \mathrm{diag}\left(\left(\frac{1-a\bar{z}}{1-\bar{a}z} \right)^{(1-2b)/2}, \left(\frac{1-a\bar{z}}{1-\bar{a}z} \right)^{-[(1+2b)/2]} \right).$$

2. Local expansions and deformation equations

Now consider multivalued solutions of the Dirac equation, which are branched at two points $a_1, a_2 \in D$ with fixed monodromies $e^{2\pi i \nu_{1,2}}$. In a sufficiently small finite neighborhood of each branch point, any such solution can be represented by an expansion of the form

$$\psi[a_j] = \sum_{n \in \mathbb{Z}+1/2} \alpha_n^j w_{n+\nu_j}(z,a_j) + \beta_n^j w_{n-\nu_j}^*(z,a_j), \quad j=1,2.$$

It is convenient to introduce instead of ν_j a new parameter $\tilde{\nu}_j$ $(j=1,2)$, which can be equal to either ν_j or ν_j+1. We assume that $-1 < \nu_{1,2} < 0$ and thus $0 < |\tilde{\nu}_{1,2}| < 1$. Consider the response functions $W_j(z,\tilde{\nu})$ and $W_j^*(z,\tilde{\nu})$ $(j=1,2)$, which satisfy the following conditions.

- $W_j(z,\tilde{\nu})$ and $W_j^*(z,\tilde{\nu})$ are multivalued solutions of the Dirac equation with the above monodromy which are square integrable (with the measure $d\mu$) as $|z| \to 1$.
- $W_j(z,\tilde{\nu})$ and $W_j^*(z,\tilde{\nu})$ have local expansions of the form

$$W_j(z,\tilde{\nu})[a_k] = \delta_{jk} w_{-1/2+\tilde{\nu}_k}(z,a_k) + \sum_{n>0} [a_{n,j}^k w_{n+\tilde{\nu}_k}(z,a_k) + b_{n,j}^k w_{n-\tilde{\nu}_k}^*(z,a_k)], \tag{3.71}$$

$$W_j^*(z,\tilde{\nu})[a_k] = \delta_{jk} w_{-1/2-\tilde{\nu}_k}^*(z,a_k) + \sum_{n>0} [c_{n,j}^k w_{n+\tilde{\nu}_k}(z,a_k) + d_{n,j}^k w_{n-\tilde{\nu}_k}^*(z,a_k)], \tag{3.72}$$

  where $k=1,2$ and $n \in \mathbb{Z}+1/2$.

It turns out that for real values of E such that $|E| < m$, these requirements fix $W_j(z,\tilde{\nu})$ and $W_j^*(z,\tilde{\nu})$ uniquely. Therefore, the expansions (3.71) and (3.72) can be thought of as defining the coefficients $a_{n,j}^k$, $b_{n,j}^k$, $c_{n,j}^k$, and $d_{n,j}^k$ as functions of a and $\tilde{\nu}$.

The lowest order coefficients satisfy a set of deformation equations in a (see Theorem 5.0 in Ref. 14). If we introduce the 2×2 matrices with the elements $A_{jk}=a_j\delta_{jk}$, $\bar{A}_{jk}=\bar{a}_j\delta_{jk}$, $\Lambda_{jk}=\tilde{\nu}_j\delta_{jk}$, and also

$$(\mathbf{a}_1)_{jk}=a^k_{1/2,j}/(1-|a_k|^2),\quad e=\Lambda-b\mathbf{1}+[\mathbf{a}_1,A],$$

$$(\mathbf{b}_1)_{jk}=b^k_{1/2,j}/(1-|a_k|^2),\quad f=A\mathbf{b}_1\bar{A}-\mathbf{b}_1,$$

$$(\mathbf{c}_1)_{jk}=c^k_{1/2,j}/(1-|a_k|^2),\quad g=\mathbf{c}_1-\bar{A}\mathbf{c}_1A,$$

$$(\mathbf{d}_1)_{jk}=d^k_{1/2,j}/(1-|a_k|^2),\quad h=\Lambda-b\mathbf{1}-[\mathbf{d}_1,\bar{A}], \tag{3.73}$$

these equations are given by

$$de=fG+Fg+[E,e], \tag{3.74}$$

$$df=eF+Fh+Ef-fH, \tag{3.75}$$

$$dg=hG+Ge+Hg-gE, \tag{3.76}$$

$$dh=gF+Gf+[H,h], \tag{3.77}$$

where

$$E=(\Lambda-(b+1/2)\mathbf{1})\frac{Ad\bar{A}+\bar{A}dA}{1-|A|^2}+[dA,\mathbf{a}_1],$$

$$F=dA\mathbf{b}_1\bar{A}+A\mathbf{b}_1d\bar{A},$$

$$G=d\bar{A}\mathbf{c}_1A+\bar{A}\mathbf{c}_1dA,$$

$$H=-(\Lambda-(b-1/2)\mathbf{1})\frac{Ad\bar{A}+\bar{A}dA}{1-|A|^2}+[d\bar{A},\mathbf{d}_1].$$

One also has symmetry relations

$$ef-fh=ge-hg=0, \tag{3.78}$$

$$e^2-fg=h^2-gf=\mu^2\mathbf{1}, \tag{3.79}$$

$$[e\sin\pi\Lambda(1-|A|^2)]^\dagger=h\sin\pi\Lambda(1-|A|^2), \tag{3.80}$$

$$[f\sin\pi\Lambda(1-|A|^2)]^\dagger=f\sin\pi\Lambda(1-|A|^2), \tag{3.81}$$

$$[g\sin\pi\Lambda(1-|A|^2)]^\dagger=g\sin\pi\Lambda(1-|A|^2). \tag{3.82}$$

In addition, the diagonal elements $a^j_{1/2,j}$ and $d^j_{1/2,j}$ may be expressed as follows:

$$a^j_{1/2,j} = \bar{a}_j m_+(\tilde{\nu}_j - 1/2, b) + \sum_{k\neq j} e_{jk} a^j_{1/2,k} + \sum_k f_{jk}\bar{a}_k c^j_{1/2,k}, \tag{3.83}$$

$$d^j_{1/2,j} = a_j m_-(-\tilde{\nu}_j - 1/2, b) - \sum_{k\neq j} h_{jk} d^j_{1/2,k} - \sum_k g_{jk} a_k b^j_{1/2,k}. \tag{3.84}$$

It is known that the system (3.74)–(3.77) combined with the relations (3.78)–(3.82) can be integrated in terms of a Painlevé VI transcendent. In particular, if we choose $a_1=0$, $a_2=\sqrt{s}$, and set

$$\frac{f_{12}f_{21}}{f_{11}f_{22}} = \frac{1-w}{1-s}, \tag{3.85}$$

then for $\tilde{\nu}_{1,2}>0$ the function $w(s)$ satisfies PVI Eq. (1.1) with parameters (1.3) (for the details of the proof, see Ref. 14).

3. Tau function

The link between the deformation equations and the tau function considered above is provided by a formula for the derivative of the Green's function of $\hat{H}^{(a,\nu)}$,

$$(1-|a_j|^2)\partial_{a_j}\ddot{G}^{(a,\nu)}(z,z') = \frac{1}{2R\sin\pi\tilde{\nu}_j} W_j(z,\tilde{\nu}) \otimes W^*_j(z',\tilde{\nu})^\dagger, \tag{3.86}$$

$$(1-|a_j|^2)\partial_{\bar{a}_j}\ddot{G}^{(a,\nu)}(z,z') = \frac{1}{2R\sin\pi\tilde{\nu}_j} W^*_j(z,\tilde{\nu}) \otimes W_j(z',\tilde{\nu})^\dagger. \tag{3.87}$$

Remarkable factorized form of these expressions follows from the fact that Green's function $\ddot{G}^{(a,\nu)}(z,z')$ inverts the operator $\hat{H}^{(a,\nu)}-E$, and from the analysis of the local expansions of $\ddot{G}^{(a,\nu)}(z,z')$ near the branch points. Numerical factors in (3.86) and (3.87) may be determined using a variant of Stokes theorem calculations from Sec. III A. Different boundary conditions for $\hat{H}^{(a,\nu)}$ are encoded into the choice of $\{\tilde{\nu}_j\}$: the value $\tilde{\nu}_j=\nu_j+1$ (or $\tilde{\nu}_j=\nu_j$) corresponds to the functions whose upper (lower) component is regular at a_j.

Using (3.86) and (3.87) one can show (see Theorem 6.3 in Ref. 14) that the logarithmic derivative of the τ-function (3.6) may be written in terms of the lowest order expansion coefficients of $W_j(z,\tilde{\nu})$ and $W^*_j(z,\tilde{\nu})$,

$$d\ln\tau(a,a^0) = \sum_j \frac{1}{1-|a_j|^2}\{a^j_{1/2,j}da_j + d^j_{1/2,j}d\bar{a}_j\}. \tag{3.88}$$

Specializing this formula to the case $a_1=0$, $a_2=\sqrt{s}$, and $\tilde{\nu}_{1,2}>0$ and using (3.83) and (3.84), PBT have obtained the relation (1.2) between the corresponding τ-function and the Painlevé VI transcendent defined by (3.85). Analogous result holds for arbitrary branch point positions, as the τ-function actually depends only on the geodesic distance between a_1 and a_2. The proof of the last statement is given in the Appendix.

E. Long-distance asymptotics

Asymptotics of the PBT τ-function as $l_s\to\infty$ (i.e., as $s\to 1$) can be obtained by expanding the Fredholm determinant (3.66),

$$\ln\tau(s) = -\sum_{n=1}^{\infty}\frac{\mathrm{Tr}(L_{\nu_2,s}L'_{\nu_1,s})^n}{n}. \tag{3.89}$$

The leading order behavior is determined by the first term in (3.89),

$$\mathrm{Tr}(L_{\nu_2,s}L'_{\nu_1,s}) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \rho(p)\rho(q)\mathcal{F}_{\nu_2}(p,q)\mathcal{F}_{\nu_1}(-p,-q)e^{i(p-q)l_s}. \tag{3.90}$$

The integrand in (3.90) has particularly simple analytical properties in the complex p-and q-planes. It has already been noted above that $\mathcal{F}_{\nu_2}(p,q)$ and $\mathcal{F}_{\nu_1}(-p,-q)$ have simple poles at $p=\pm i(1+2\mu+2n_1)$ and $q=\pm i(1+2\mu+2n_2)$, where $n_{1,2}=0,1,2,\ldots$. At the same points, the functions $\rho(p)$ and $\rho(q)$ have simple zeros and thus the whole integrand has simple poles. We can then close the contours of integration over p and q in the upper and lower half-plane, respectively (recall that $l_s>0$). Summing over the residues, one finds a long-distance expansion of the integral (3.90). The leading term of this expansion corresponds to the poles at $p=i(1+2\mu)$ and $q=-i(1+2\mu)$ and yields

$$\mathrm{Tr}(L_{\nu_2,s}L'_{\nu_1,s}) \simeq \pi^2 2^{4\mu}[\mathrm{res}\,\mathcal{F}_{\nu_1}(p,q)]_{\substack{p=i(1+2\mu)\\ q=-i(1+2\mu)}}[\mathrm{res}\,\mathcal{F}_{\nu_2}(p,q)]_{\substack{p=i(1+2\mu)\\ q=-i(1+2\mu)}} e^{-2(1+2\mu)l_s} + O(e^{-2(2+2\mu)l_s}). \tag{3.91}$$

The residues in the last formula can be extracted from, e.g., the representation (3.64). After somewhat cumbersome computation, one finds

$$[\mathrm{res}\,\mathcal{F}_{\nu}(p,q)]_{\substack{p=i(1+2\mu)\\ q=-i(1+2\mu)}} = \frac{2\sin\pi\nu}{\pi^2}\frac{\Gamma(\mu+2+\nu-b)\Gamma(\mu-\nu+b)}{\Gamma(2+2\mu)}. \tag{3.92}$$

Combining (3.89)–(3.92) and using that $4e^{-2l_s}\simeq 1-s$ as $s\to 1$, we finally obtain the asymptotics

$$\tau(s) \simeq 1 - A_\tau(1-s)^{1+2\mu} + O((1-s)^{2+2\mu}), \quad \text{as} \quad s\to 1,$$

with A_τ given by (1.5). This finishes the proof of Theorem 1.1.

IV. *PVI*→*PV*: FLAT SPACE LIMIT

In the present section, the analogs of the above results in the limit of flat space are established. Since the SAEs and the spectrum of the corresponding Dirac Hamiltonian have already been discussed in Ref. 22 (see also Ref. 23 for the zero-field case), we will not dwell much on this point. One-vortex Green's function was also computed in a closed form by Gavrilov *et al.*[24] However, the representation found in Ref. 24 is inconvenient for our purposes, so below we obtain another formula, in which the vortex-dependent contribution to the resolvent is manifestly separated from the "free" part. This formula enables us to find Fredholm determinant representations for the two-point tau function of the Dirac Hamiltonian on the plane, which turns out to be related to a class of Painlevé V transcendents.

A. One-vortex Green's function

Quantum motion of a Dirac particle on the plane in the presence of an external magnetic field is described by the Hamiltonian,

$$\hat{H} = \begin{pmatrix} m & 2D_z \\ -2D_{\bar{z}} & -m \end{pmatrix}, \quad z\in\mathbb{C}.$$

As above, we will consider a vector potential describing the superposition of a uniform magnetic field B and of the field of an AB flux $\Phi=2\pi\nu$ $(-1<\nu<0)$, situated at the origin. Analogous to (2.6) and (2.7), one can choose

$$\mathcal{A}^{(B)} = -\frac{i}{4}B(\bar{z}dz - zd\bar{z}) = \frac{Br^2}{2}d\varphi, \tag{4.1}$$

$$\mathcal{A}^{(\nu)} = -\frac{i\nu}{2}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right) = \nu d\varphi, \tag{4.2}$$

and then the Hamiltonian becomes

$$\hat{H} = \begin{pmatrix} m & 2\partial_z + \frac{B\bar{z}}{2} + \frac{\nu}{z} \\ -2\partial_{\bar{z}} + \frac{Bz}{2} + \frac{\nu}{\bar{z}} & -m \end{pmatrix}. \tag{4.3}$$

Radial Hamiltonians $\hat{H}_{l_0+\nu}$ $(l_0 \in \mathbb{Z})$, corresponding to different angular momentum eigenvalues (equal to $l_0 + 1/2$), are defined by

$$\hat{H}_l = \begin{pmatrix} m & \partial_r + \frac{l+1}{r} + \frac{Br}{2} \\ -\partial_r + \frac{l}{r} + \frac{Br}{2} & -m \end{pmatrix}.$$

The details of further calculation depend on the sign of B and below we consider different cases separately.

1. $B>0$

First we introduce two families of solutions of the Dirac equation without AB field, $(\hat{H}^{(0)} - E)\psi = 0$,

$$\Psi_+(z,\theta) = e^{-(B|z|^2/4) - \sqrt{B/2}\bar{z}e^{\theta}} \begin{pmatrix} C_+^{-1} e^{-\theta/2}(1 + \sqrt{B/2}ze^{-\theta})^{-1-(\lambda^2/2B)} \\ C_+ e^{\theta/2}(1 + \sqrt{B/2}ze^{-\theta})^{-(\lambda^2/2B)} \end{pmatrix}, \tag{4.4}$$

$$\Psi_-(z,\theta) = e^{(B|z|^2/4) + \sqrt{B/2}ze^{-\theta}} \begin{pmatrix} C_-^{-1} e^{-\theta/2}(1 + \sqrt{B/2}\bar{z}e^{\theta})^{\lambda^2/2B} \\ -C_- e^{\theta/2}(1 + \sqrt{B/2}\bar{z}e^{\theta})^{-1+(\lambda^2/2B)} \end{pmatrix}, \tag{4.5}$$

with

$$\lambda = \sqrt{m^2 - E^2},$$

$$C_\pm = \left(\frac{m-E}{m+E}\right)^{1/4} \left(\frac{2B}{\lambda^2}\right)^{\pm 1/4}. \tag{4.6}$$

Define $\hat{\Psi}_\pm(z,\theta) = \Psi_\pm(z \leftrightarrow \bar{z}, \theta \leftrightarrow -\theta)$, as in (2.31). The functions $\Psi_+(z,\theta)$ and $\hat{\Psi}_-(z,\theta)$ are delimited by the horizontal branch cuts $(-\infty + i(\varphi + \pi + 2\pi\mathbb{Z}), \ln(\sqrt{B/2}r) + i(\varphi + \pi + 2\pi\mathbb{Z})]$ in the θ-plane, whereas the branch cuts for $\Psi_-(z,\theta)$ and $\hat{\Psi}_+(z,\theta)$ are $[-\ln(\sqrt{B/2}r) + i(\varphi + \pi + 2\pi\mathbb{Z}), \infty + i(\varphi + \pi + 2\pi\mathbb{Z}))$ (see Fig. 5). The arguments of both $1 + \sqrt{B/2}ze^{-\theta}$ and $1 + \sqrt{B/2}\bar{z}e^{\theta}$ are fixed to be zero on the line $\mathrm{Im}\ \varphi = \pi$.

Next introduce the following functions:

$$\mathrm{w}_l^{(\mathrm{I})}(z) = \int_{C_0(z)} e^{(l+1/2)\theta} \Psi_-(z,\theta) d\theta = e^{i\pi l} \begin{pmatrix} e^{il\varphi} & 0 \\ 0 & e^{i(l+1)\varphi} \end{pmatrix} w_l^{(\mathrm{I})}(r), \tag{4.7}$$

$$\mathrm{w}_l^{(\mathrm{II},\pm)}(z) = \pm \int_{C_\pm(z)} e^{(l+1/2)\theta} \Psi_+(z,\theta) d\theta = 2\pi i e^{i\pi l} \begin{pmatrix} e^{il\varphi} & 0 \\ 0 & e^{i(l+1)\varphi} \end{pmatrix} w_l^{(\mathrm{II},\pm)}(r), \tag{4.8}$$

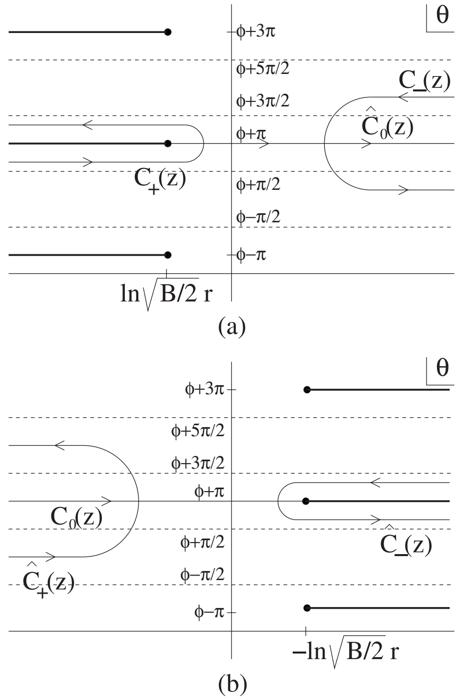

FIG. 5. Branch cuts and integration contours in the θ-plane for (a) $\Psi_+(z,\theta)$ and $\hat{\Psi}_-(z,\theta)$ and (b) $\Psi_-(z,\theta)$ and $\hat{\Psi}_+(z,\theta)$.

$$\hat{\mathrm{w}}_l^{(\mathrm{I})}(z) = \int_{\hat{C}_0(z)} e^{-(l+1/2)\theta}\hat{\Psi}_-(z,\theta)d\theta = e^{-i\pi l}\begin{pmatrix} e^{-il\varphi} & 0 \\ 0 & e^{-i(l+1)\varphi}\end{pmatrix} w_l^{(\mathrm{I})}(r), \tag{4.9}$$

$$\hat{\mathrm{w}}_l^{(\mathrm{II},\pm)}(z) = \mp\int_{\hat{C}_\mp(z)} e^{-(l+1/2)\theta}\hat{\Psi}_+(z,\theta)d\theta = 2\pi i e^{-i\pi l}\begin{pmatrix} e^{-il\varphi} & 0 \\ 0 & e^{-i(l+1)\varphi}\end{pmatrix} w_l^{(\mathrm{II},\pm)}(r), \tag{4.10}$$

where the integration contours $C_0(z)$, $\hat{C}_0(z)$, $C_\pm(z)$, and $\hat{C}_\pm(z)$ are shown in Fig. 5. Notice that $\mathrm{w}_l^{(\mathrm{II},+)}(z)$ and $\hat{\mathrm{w}}_l^{(\mathrm{II},+)}(z)$ are well-defined for $l>-1$, while $\mathrm{w}_l^{(\mathrm{II},-)}(z)$, $\hat{\mathrm{w}}_l^{(\mathrm{II},-)}(z)$ are well-defined for $l<0$. The functions $w_l^{(\mathrm{I})}(r)$ and $w_l^{(\mathrm{II},\pm)}(r)$ satisfy radial Dirac equation $(\hat{H}_l-E)w_l=0$. They are explicitly given by

$$w_l^{(\mathrm{I})}(r) = \frac{\lambda}{\sqrt{2B}}\Gamma\left(\frac{\lambda^2}{2B}\right)e^{-Br^2/4}\begin{pmatrix} C_+^{-1}(\sqrt{B/2}r)^{-l}U\left(\frac{\lambda^2}{2B}+1,1-l,\frac{Br^2}{2}\right) \\ C_+(\sqrt{B/2}r)^{-l-1}U\left(\frac{\lambda^2}{2B},-l,\frac{Br^2}{2}\right)\end{pmatrix} \tag{4.11}$$

$$= \frac{\lambda}{\sqrt{2B}}\Gamma\left(\frac{\lambda^2}{2B}\right)e^{-Br^2/4}\begin{pmatrix} C_+^{-1}(\sqrt{B/2}r)^{l}U\left(\frac{\lambda^2}{2B}+1+l,1+l,\frac{Br^2}{2}\right) \\ C_+(\sqrt{B/2}r)^{l+1}U\left(\frac{\lambda^2}{2B}+1+l,2+l,\frac{Br^2}{2}\right)\end{pmatrix}, \tag{4.12}$$

$$w_l^{(\mathrm{II},+)}(r) = \begin{pmatrix} \dfrac{\Gamma\left(\frac{\lambda^2}{2B}+1+l\right)}{\Gamma\left(\frac{\lambda^2}{2B}+1\right)\Gamma(1+l)} C_+^{-1} e^{-Br^2/4}(\sqrt{B/2}r)^l M\left(\frac{\lambda^2}{2B}+1+l,1+l,\frac{Br^2}{2}\right) \\ -\dfrac{\Gamma\left(\frac{\lambda^2}{2B}+1+l\right)}{\Gamma\left(\frac{\lambda^2}{2B}\right)\Gamma(2+l)} C_+ e^{-Br^2/4}(\sqrt{B/2}r)^{l+1} M\left(\frac{\lambda^2}{2B}+1+l,2+l,\frac{Br^2}{2}\right) \end{pmatrix}, \tag{4.13}$$

$$w_l^{(\mathrm{II},-)}(r) = \begin{pmatrix} \dfrac{C_+^{-1} e^{-Br^2/4}}{\Gamma(1-l)} (\sqrt{B/2}r)^{-l} M\left(\frac{\lambda^2}{2B}+1,1-l,\frac{Br^2}{2}\right) \\ -\dfrac{C_+ e^{-Br^2/4}}{\Gamma(-l)} (\sqrt{B/2}r)^{-l-1} M\left(\frac{\lambda^2}{2B},-l,\frac{Br^2}{2}\right) \end{pmatrix}, \tag{4.14}$$

where $M(\alpha,\beta,s)$ and $U(\alpha,\beta,s)$ denote Kummer's functions (see Ref. 25). The solutions (4.11)–(4.14) have the same integrability properties, as in the case of the disk. Namely,

- $w_l^{(\mathrm{I})}(r)$ is the only solution of the radial Dirac equation, square integrable as $r\to\infty$ with respect to the measure $d\mu_r = r dr$.
- For $l\geq 0$ ($l\leq -1$), $w_l^{(\mathrm{II},+)}(r)$ $[w_l^{(\mathrm{II},-)}(r)]$ is the only solution, square integrable as $r\to 0$. For $l \in (-1,0)$ both $w_l^{(\mathrm{II},\pm)}(r)$ are square integrable as $r\to 0$.
- $w_l^{(\mathrm{I})}(r)$ and $w_l^{(\mathrm{II},+)}(r)$ are linearly independent for $l>-1$ and $w_l^{(\mathrm{I})}(r)$ and $w_l^{(\mathrm{II},-)}(r)$ are linearly independent for $l<0$. The determinant of the fundamental matrix of solutions is given by

$$\det(w_l^{(\mathrm{I})}(r), w_l^{(\mathrm{II},\pm)}(r)) = -\frac{2}{\lambda r}. \tag{4.15}$$

This implies the following.

Proposition 4.1: (Reference 22) *Let* dom $\hat{H}_l = C_0^\infty(\mathbb{R}^+)$. *Then for* $l \in (-\infty,-1]\cup[0,\infty)$ $\hat{H}_l$ *is essentially self-adjoint. In the case* $l\in(-1,0)$, *it admits a one-parameter family of SAEs.*

We will consider two particular SAEs of $\hat{H}_\nu$, whose domains are composed of functions with regular at $r=0$ lower (or upper) component. Similar to the above, these two cases will be referred to as corresponding to the SAE parameter $\Theta=\pi/2$ ($-\pi/2$).

Using (4.15), one may show that for $l\in(-\infty,-1]\cup[0,\infty)$ the radial Green's function $G_{E,l}(r,r')$ is given by

$$G_{E,l}(r,r') = \begin{cases} \dfrac{\lambda}{2} w_l^{(\mathrm{II},\pm)}(r) \otimes (w_l^{(\mathrm{I})}(r'))^T & \text{for} \quad 0<r<r'<1 \\ \dfrac{\lambda}{2} w_l^{(\mathrm{I})}(r) \otimes (w_l^{(\mathrm{II},\pm)}(r'))^T & \text{for} \quad 0<r'<r<1. \end{cases} \tag{4.16}$$

As in Sec. II C, the sign + (−) in (4.16) corresponds to $l\geq 0$ ($l\leq -1$). For $l\in(-1,0)$, the relation (4.16) gives the Green's function of the SAE with $\Theta=-\pi/2$ (when taken with the sign −) and with $\Theta=\pi/2$ (for the sign +).

We can now repeat word-for-word the calculation of the full one-vortex Green's function $G(z,z')$ from Sec. II E, using the formula (4.16) and contour integral representations (4.7)–(4.10). Setting for definiteness $\Theta=-\pi/2$, one obtains the following representation for the Green's function of the Dirac Hamiltonian (4.3):

$$G(z,z') = \begin{cases} e^{-i\nu(\varphi-\varphi'+2\pi)}G^{(0)}(z,z') + \Delta(z,z') & \text{for} \quad \varphi-\varphi' \in (-2\pi,-\pi) \\ e^{-i\nu(\varphi-\varphi')}G^{(0)}(z,z') + \Delta(z,z') & \text{for} \quad \varphi-\varphi' \in (-\pi,\pi) \\ e^{-i\nu(\varphi-\varphi'-2\pi)}G^{(0)}(z,z') + \Delta(z,z') & \text{for} \quad \varphi-\varphi' \in (\pi,2\pi), \end{cases} \tag{4.17}$$

with

$$G^{(0)}(z,z') = \frac{\lambda}{4\pi}\int_{C_0(z)} d\theta \;\; \Psi_-(z,\theta) \otimes \hat{\Psi}_+^T(z',\theta), \tag{4.18}$$

$$\Delta(z,z') = \lambda e^{-i\nu(\varphi-\varphi')}\frac{1-e^{-2\pi i\nu}}{8i\pi^2}\int_{C_0(z)} d\theta_1 \int_{\mathrm{Im}\;\theta_2=\varphi'} d\theta_2 \Psi_-(z,\theta_1) \otimes \hat{\Psi}_+^T(z',\theta_2)\frac{e^{(1+\nu+1/2)(\theta_1-\theta_2)}}{e^{\theta_1-\theta_2}-1}. \tag{4.19}$$

The integrals (4.18) and (4.19) reduce to

$$G^{(0)}(z,z') = e^{(B/4)(\bar{z}z'-z\bar{z}')}\begin{pmatrix} \zeta_{11}(u(z,z')) & \dfrac{\sqrt{u(z,z')}}{z'-z}\zeta_{12}(u(z,z')) \\ -\dfrac{\sqrt{u(z,z')}}{\bar{z}'-\bar{z}}\zeta_{21}(u(z,z')) & -\zeta_{22}(u(z,z')) \end{pmatrix}, \tag{4.20}$$

$$\Delta(z,z') = \frac{\sin\,\pi\nu}{\pi}\int_{-\infty}^{\infty} d\theta \frac{e^{(1+\nu)\theta+i(\varphi-\varphi')}}{e^{\theta+i(\varphi-\varphi')}+1}e^{-(B/2)rr'\,\sinh\,\theta} \times \begin{pmatrix} \zeta_{11}(v(r,r',\theta)) & \dfrac{e^{-i\varphi'}\sqrt{v(r,r',\theta)}}{re^{-\theta}+r'}\zeta_{12}(v(r,r',\theta)) \\ \dfrac{e^{\theta+i\varphi}\sqrt{v(r,r',\theta)}}{re^{\theta}+r'}\zeta_{21}(v(r,r',\theta)) & e^{\theta+i(\varphi-\varphi')}\zeta_{22}(v(r,r',\theta)) \end{pmatrix}, \tag{4.21}$$

where $\zeta(u)$, $u(z,z')$, and $v(r,r',\theta)$ are defined as follows:

$$\zeta(u) = \frac{e^{-(Bu/4)}}{2\pi}\sqrt{\frac{B}{2}}\Gamma\left(\frac{\lambda^2}{2B}+1\right)\begin{pmatrix} C_+^{-2}U\left(\dfrac{\lambda^2}{2B}+1,1,\dfrac{Bu}{2}\right) & \sqrt{\dfrac{Bu}{2}}U\left(\dfrac{\lambda^2}{2B}+1,2,\dfrac{Bu}{2}\right) \\ \sqrt{\dfrac{Bu}{2}}U\left(\dfrac{\lambda^2}{2B}+1,2,\dfrac{Bu}{2}\right) & C_+^2U\left(\dfrac{\lambda^2}{2B},1,\dfrac{Bu}{2}\right) \end{pmatrix}, \tag{4.22}$$

$$u(z,z') = |z-z'|^2, \quad v(r,r',\theta) = r^2 + r'^2 + 2rr'\,\cosh\,\theta. \tag{4.23}$$

Remark: Green's function for $\Theta=\pi/2$ can be computed in a completely analogous manner. Final answer differs from (4.17) and (4.20)–(4.23) in only one point: one has to replace $e^{(1+\nu)\theta+i(\varphi-\varphi')}$ by $-e^{\nu\theta}$ in the first line of the integral representation (4.21) for $\Delta(z,z')$, just as in (2.49) versus (2.47) in the disk case.

2. $B<0$

In Sec. IV A 1, we have obtained $G(z,z')$ imitating the calculation made on the Poincaré disk. Alternatively, one can simply consider the limit

$$R \to \infty, \quad r \to \frac{r}{R}, \quad r' \to \frac{r'}{R}$$

of the relations (2.40) and (2.45)–(2.48) using asymptotic properties of hypergeometric functions. For $B<0$ and $\Theta=-\pi/2$ (regular upper component), this gives a representation of the one-vortex Green's function, which has exactly the same form as (4.17), (4.20), and (4.21) but with

$$\zeta(u) = \frac{e^{-|B|u/4}}{2\pi}\sqrt{\frac{|B|}{2}}\Gamma\left(\frac{\lambda^2}{2|B|}+1\right)\begin{pmatrix} C_+^{-2}U\left(\frac{\lambda^2}{2|B|},1,\frac{|B|u}{2}\right) & \sqrt{\frac{|B|u}{2}}U\left(\frac{\lambda^2}{2|B|}+1,2,\frac{|B|u}{2}\right) \\ \sqrt{\frac{|B|u}{2}}U\left(\frac{\lambda^2}{2|B|}+1,2,\frac{|B|u}{2}\right) & C_+^2U\left(\frac{\lambda^2}{2|B|}+1,1,\frac{|B|u}{2}\right) \end{pmatrix} \tag{4.24}$$

and

$$C_+ = \left(\frac{m-E}{m+E}\right)^{1/4}\left(\frac{\lambda^2}{2|B|}\right)^{1/4}. \tag{4.25}$$

3. B=0

Taking further limit $B\to 0$ in (4.20)–(4.22), we get (still for $\Theta=-\pi/2$, the modification for $\Theta=\pi/2$ is as described above)

$$G^{(0)}(z,z') = \begin{pmatrix} \zeta_{11}(u(z,z')) & \frac{\sqrt{u(z,z')}}{z'-z}\zeta_{12}(u(z,z')) \\ -\frac{\sqrt{u(z,z')}}{\bar{z}'-\bar{z}}\zeta_{21}(u(z,z')) & -\zeta_{22}(u(z,z')) \end{pmatrix}, \tag{4.26}$$

$$\Delta(z,z') = \frac{\sin \pi\nu}{\pi}\int_{-\infty}^{\infty} d\theta \frac{e^{(1+\nu)\theta+i(\varphi-\varphi')}}{e^{\theta+i(\varphi-\varphi')}+1} \times \begin{pmatrix} \zeta_{11}(v(r,r',\theta)) & \frac{e^{-i\varphi'}\sqrt{v(r,r',\theta)}}{re^{-\theta}+r'}\zeta_{12}(v(r,r',\theta)) \\ \frac{e^{\theta+i\varphi}\sqrt{v(r,r',\theta)}}{re^{\theta}+r'}\zeta_{21}(v(r,r',\theta)) & e^{\theta+i(\varphi-\varphi')}\zeta_{22}(v(r,r',\theta)) \end{pmatrix}, \tag{4.27}$$

where the matrix $\zeta(u)$ is given by

$$\zeta(u) = \frac{\lambda}{2\pi}\begin{pmatrix} C_+^{-2}K_0(\lambda\sqrt{u}) & K_1(\lambda\sqrt{u}) \\ K_1(\lambda\sqrt{u}) & C_+^2K_0(\lambda\sqrt{u}) \end{pmatrix}, \quad C_+ = \left(\frac{m-E}{m+E}\right)^{1/4}, \tag{4.28}$$

and $K_{0,1}(s)$ denote the modified Bessel functions.

Using integral representations for $K_{0,1}(s)$, one can write (4.27) in a different form. Namely, for any $\alpha\in\mathbb{R}$ such that $|\varphi-\alpha|<\pi/2$, $|\varphi'-\alpha|<\pi/2$, we have

$$\Delta(z,z') = e^{-i\nu(\varphi-\varphi')}\frac{\lambda \sin \pi\nu}{4\pi^2}\int_{-\infty}^{\infty} d\theta_1\int_{-\infty}^{\infty} d\theta_2 \; e^{-\lambda r \cosh(\theta_1+i(\alpha-\varphi))-\lambda r' \cosh(\theta_2+i(\alpha-\varphi'))} \times \frac{e^{(3/2+\nu)(\theta_1-\theta_2)}}{e^{\theta_1-\theta_2}+1}\begin{pmatrix} C_+^{-1}e^{-(\theta_1+i\alpha)/2} \\ C_+e^{(\theta_1+i\alpha)/2} \end{pmatrix} \otimes \begin{pmatrix} C_+^{-1}e^{(\theta_2+i\alpha)/2} \\ C_+e^{-(\theta_2+i\alpha)/2} \end{pmatrix}^T.$$

Example: For $0<\varphi<\pi$ and $0<\varphi'<\pi$, we may take $\alpha=\pi/2$ so that

$$\Delta(z,z') = e^{-i\nu(\varphi-\varphi')}\frac{\lambda \sin \pi\nu}{4\pi^2}\int_{-\infty}^{\infty} d\theta_1 \int_{-\infty}^{\infty} d\theta_2 \; e^{-\lambda(y\cosh\theta_1 + ix\sinh\theta_1)-\lambda(y'\cosh\theta_2+ix'\sinh\theta_2)} \times \frac{e^{(3/2+\nu)(\theta_1-\theta_2)}}{e^{\theta_1-\theta_2}+1}\begin{pmatrix} C_+^{-1}e^{-\theta_1/2} \\ iC_+e^{\theta_1/2}\end{pmatrix} \otimes \begin{pmatrix} C_+^{-1}e^{\theta_2/2} \\ -iC_+e^{-\theta_2/2}\end{pmatrix}^T. \tag{4.29}$$

As we will see a bit later, this formula makes the computation of zero field form factors particularly simple. Note that similar expressions for the one-vortex Green's function on the plane have already appeared in different papers (see, e.g., Refs. 4 and 26).

B. Two-point tau function and Painlevé V

In order to write the tau function as a Fredholm determinant, we will consider free Dirac equation in another gauge. Set the potential of the uniform magnetic field to be

$$\mathcal{A}^{(B)} = -Bydx, \tag{4.30}$$

so that the corresponding Dirac Hamiltonian

$$\hat{H}_{\text{tr}}^{(0)} = \begin{pmatrix} m & \partial_x - i\partial_y - iBy \\ -\partial_x - i\partial_y + iBy & -m \end{pmatrix}$$

commutes with the x-momentum operator $\hat{P}_x = -i\partial_x$. The eigenspace of $\hat{P}_x$ with momentum p is spanned by the functions $g(p,y)e^{ipx}$. Let us look at the solutions of the partial Dirac equation

$$(\hat{H}_p - E)g(p,y) = 0, \quad \hat{H}_p = \begin{pmatrix} m & -i(\partial_y - p + By) \\ -i(\partial_y + p - By) & -m \end{pmatrix}. \tag{4.31}$$

As above, we assume that E is real and $|E| < m$. It is convenient to choose two linearly independent solutions of (4.31) as follows:

$$B>0: \quad \Phi^{(\pm)}(p,y) = \left[\frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{\lambda^2}{2B}+1\right)\right]^{1/2}\begin{pmatrix} C_+^{-1}D_{-(\lambda^2/2B)-1}\left(\pm\sqrt{2B}\left(y-\frac{p}{B}\right)\right) \\ \pm iC_+D_{-(\lambda^2/2B)}\left(\pm\sqrt{2B}\left(y-\frac{p}{B}\right)\right)\end{pmatrix}, \tag{4.32}$$

$$B<0: \quad \Phi^{(\pm)}(p,y) = \left[\frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{\lambda^2}{2|B|}+1\right)\right]^{1/2}\begin{pmatrix} C_+^{-1}D_{-(\lambda^2/2|B|)}\left(\pm\sqrt{2|B|}\left(y+\frac{p}{|B|}\right)\right) \\ \pm iC_+D_{-(\lambda^2/2|B|)-1}\left(\pm\sqrt{2|B|}\left(y+\frac{p}{|B|}\right)\right)\end{pmatrix}, \tag{4.33}$$

$$B=0: \quad \Phi^{(\pm)}(p,y) = \frac{e^{\mp\sqrt{\lambda^2+p^2}y}}{\sqrt{2}}\begin{pmatrix} C_+^{-1}\left(1 \pm \frac{p}{\sqrt{\lambda^2+p^2}}\right)^{1/2} \\ \pm iC_+\left(1 \mp \frac{p}{\sqrt{\lambda^2+p^2}}\right)^{1/2}\end{pmatrix}. \tag{4.34}$$

Here, $D_\alpha(s)$ denotes the parabolic cylinder function and the constant C_+ in (4.32)–(4.34) is determined by (4.6), (4.25), and (4.28), correspondingly. Note that $\Phi^{(+)}(p,y)[\Phi^{(-)}(p,y)]$ is square integrable as $y \to \infty [y \to -\infty]$. These two solutions satisfy symmetry relations analogous to (3.39)

$$\Phi^{(+)}(p,y) = \sigma_z \Phi^{(-)}(-p,-y), \quad \Phi^{(\pm)}(p,y) = \sigma_z \overline{\Phi^{(\pm)}(p,y)}. \tag{4.35}$$

The normalization in (4.32)–(4.34) was chosen so that in all three cases

$$\det(\Phi^{(+)}(p,y),\Phi^{(-)}(p,y)) = -i. \tag{4.36}$$

We now adapt the reasoning of Sec. III C 1 to flat space. Consider a line $\mathcal{L}_{y^{(0)}}=\{(x,y)\in\mathbb{R}^2 | y=y^{(0)}\}$ and an arbitrary $\mathbb{C}^2$-valued function $g_{y^{(0)}}(x)\in H^{1/2}(\mathcal{L}_{y^{(0)}})$, written as Fourier integral

$$g_{y^{(0)}}(x) = \int_{-\infty}^{\infty} dp \;\; g(p,y^{(0)}) e^{ipx}. \tag{4.37}$$

Decompose Fourier transform $g(p,y^{(0)})$ as follows:

$$g(p,y^{(0)}) = \tilde{g}_+(p,y^{(0)})\Phi^{(+)}(p,y^{(0)}) + \tilde{g}_-(p,y^{(0)})\Phi^{(-)}(p,y^{(0)}),$$

where $\Phi^{(\pm)}(p,y^{(0)})$ denote the functions defined by (4.32), (4.33), or (4.34), depending on the value of B. The formula (4.36) and symmetry relations (4.35) imply that

$$\tilde{g}_\pm(p,y^{(0)}) = \mp i(\Phi^{(\mp)}(p,y^{(0)}))^\dagger \sigma_x g(p,y^{(0)}). \tag{4.38}$$

Recall that $\tilde{g}_+(p,y^{(0)})$ and $\tilde{g}_-(p,y^{(0)})$ can be thought of as coordinates in the spaces of boundary values of solutions of the free Dirac equation $(\hat{H}_{\mathrm{tr}}^{(0)}-E)\psi=0$ in the half planes $y>y^{(0)}$ and $y<y^{(0)}$.

It is now straightforward to write down the analogs of Propositions 3.8 and 3.9.

Proposition 4.2: *Let us consider a strip $\mathcal{S}=\{(x,y)\in\mathbb{R}^2 | y^{(L)}<y<y^{(R)}\}$. Suppose that $\psi\in H^{1/2}(\partial\mathcal{S})$ can be continued to $\mathcal{S}$ as a solution of the free Dirac equation $(\hat{H}_{\mathrm{tr}}^{(0)}-E)\psi=0$. Then,*

$$\begin{pmatrix} \tilde{\psi}_{L,-}(p) \\ \tilde{\psi}_{R,+}(p) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\psi}_{L,+}(p) \\ \tilde{\psi}_{R,-}(p) \end{pmatrix}. \tag{4.39}$$

Proposition 4.3: *Assume that the strip $\mathcal{S}$ contains one branching point a_0 (i.e., $y^{(L)}<a_{0y}<y^{(R)}$) and introduce a horizontal branch cut $\ell=(-\infty+ia_{0y},a_{0x}+ia_{0y}]$. Suppose that $\psi\in H^{1/2}(\partial\mathcal{S})$ is the boundary value of a multivalued solution of the free Dirac equation on $\mathcal{S}\backslash\ell$, which is characterized by the monodromy $e^{2\pi i\nu}$ at the point a_0. Then,*

$$\begin{pmatrix} \tilde{\psi}_{L,-}(p) \\ \tilde{\psi}_{R,+}(p) \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_{\mathcal{S}}(a_0) & \hat{\beta}_{\mathcal{S}}(a_0) \\ \hat{\gamma}_{\mathcal{S}}(a_0) & \hat{\delta}_{\mathcal{S}}(a_0) \end{pmatrix} \begin{pmatrix} \tilde{\psi}_{L,+}(p) \\ \tilde{\psi}_{R,-}(p) \end{pmatrix},$$

where

$$(\hat{\alpha}_{\mathcal{S}}(a_0)\tilde{\psi}_{L,+})(p) = \int_{-\infty}^{\infty} \dot{\Delta}_-^{(a_0,\nu)}(p,q)\tilde{\psi}_{L,+}(q)dq, \tag{4.40}$$

$$(\hat{\beta}_{\mathcal{S}}(a_0)\tilde{\psi}_{R,-})(p) = \int_{-\infty}^{\infty} \dot{G}_+^{(a_0,\nu)}(p,q)\tilde{\psi}_{R,-}(q)dq, \tag{4.41}$$

$$(\hat{\gamma}_{\mathcal{S}}(a_0)\tilde{\psi}_{L,+})(p) = \int_{-\infty}^{\infty} \dot{G}_-^{(a_0,\nu)}(p,q)\tilde{\psi}_{L,+}(q)dq, \tag{4.42}$$

$$(\hat{\delta}_{\mathcal{S}}(a_0)\tilde{\psi}_{R,-})(p) = \int_{-\infty}^{\infty} \dot{\Delta}_+^{(a_0,\nu)}(p,q)\tilde{\psi}_{R,-}(q)dq, \tag{4.43}$$

and

$$\dot{\Delta}_{\pm}^{(a_0,\nu)}(p,q)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dxdx'e^{-ipx+iqx'}(\Phi^{(\mp)}(p,y))^{\dagger}\sigma_x\dot{\Delta}_{\text{tr}}^{(a_0,\nu)}(z,z')\big|_{y,y'\gtrless a_{0y}}\sigma_x\Phi^{(\mp)}(q,y'), \tag{4.44}$$

$$\dot{G}_{\pm}^{(a_0,\nu)}(p,q)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dxdx'e^{-ipx+iqx'}(\Phi^{(\pm)}(p,y))^{\dagger}\sigma_x\dot{G}_{tr}^{(a_0,\nu)}(z,z')\big|_{y\lessgtr a_{0y},y'\gtrless a_{0y}}\sigma_x\Phi^{(\mp)}(q,y'). \tag{4.45}$$

Here, $\dot{G}_{\text{tr}}^{(a_0,\nu)}(z,z')$ denotes the Green's function of the Dirac Hamiltonian $\hat{H}_{\text{tr}}^{(0)}$ on the plane with one branching point a_0 and $\dot{\Delta}_{\text{tr}}^{(a_0,\nu)}(z,z')=\dot{G}_{\text{tr}}^{(a_0,\nu)}(z,z')-G_{\text{tr}}^{(0)}(z,z')$.

The definition of the tau function of the Dirac Hamiltonian on the plane with two branch points a_1 and a_2 is also completely analogous to the disk case. Repeating the arguments of Sec. III A, one ends up with the following Fredholm determinant representation:

$$\tau(a)=\det(1-\hat{\alpha}(a_2)\hat{\delta}(a_1)), \tag{4.46}$$

where $\hat{\alpha}(a_2)$ and $\hat{\delta}(a_1)$ are given by (4.40) and (4.43). As above, the fact that the tau function depends only on the distance between the points a_1 and a_2 allows us to choose $a_1=0$, $a_2=t+i0$ ($t\in\mathbb{R}^+$), and the invariance of $\hat{H}_{\text{tr}}^{(0)}$ with respect to x-translations reduces the problem of calculation of $\tau(a)$ to finding the form factors $\dot{\Delta}_{\pm}^{(0,\nu)}(p,q)$. Finally, the symmetry of the free Dirac Hamiltonian $\hat{H}_{\text{tr}}^{(0)}$ combined with the relations (4.35) implies that

$$\dot{\Delta}_{\pm}^{(0,\nu)}(p,q)=\overline{\dot{\Delta}_{\pm}^{(0,\nu)}(p,q)}=\dot{\Delta}_{\pm}^{(0,\nu)}(q,p)=\dot{\Delta}_{\mp}^{(0,\nu)}(-p,-q). \tag{4.47}$$

The form factors $\dot{\Delta}_{\pm}^{(0,\nu)}(p,q)$ are determined by the relation (4.44) or by the equivalent formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dx\ \ dx'\ \ e^{-ipx+iqx'}\dot{\Delta}_{tr}^{(0,\nu)}(z,z')\big|_{y,y'\gtrless 0}=\dot{\Delta}_{\pm}^{(0,\nu)}(p,q)\Phi^{(\pm)}(p,y)\otimes(\Phi^{(\pm)}(q,y'))^{\dagger}. \tag{4.48}$$

We remark that $\dot{\Delta}_{\text{tr}}^{(0,\nu)}(z,z')$ and the function $\Delta(z,z')$ defined by (4.21)–(4.25) (for $B\neq 0$) or by (4.27) and (4.28) (for $B=0$) are related by

$$\dot{\Delta}_{\text{tr}}^{(0,\nu)}(z,z')=e^{i\nu(\varphi-\varphi')}\times e^{iB/2(xy-x'y')}\times\Delta(z,z'),\quad \varphi\in(-\pi,\pi). \tag{4.49}$$

Here, the first factor corresponds to a singular gauge transformation removing the AB field, and the second one comes from the change of the vector potential of the uniform magnetic field from (4.1) to (4.30).

Also note that y and y' in (4.44) and (4.48) can be chosen arbitrarily. Analogous observation in the disk case allowed us to obtain a more explicit representation for $\dot{\Delta}_{\pm}^{(0,\nu)}(p,q)$ by analyzing the asymptotics of a relation similar to (4.48) near the disk boundary. For $B\neq 0$ the asymptotic analysis of the left hand side of (4.48) as $y,y'\to\pm\infty$ becomes rather complicated and we have not managed to repeat the above trick in this case. However, when both p and q are positive or negative, one can choose y and y' in such a way that the arguments of parabolic cylinder functions in the right hand side of one of the relations (4.48) are equal to zero. This leads to a simpler [than (4.48) for general y,y'] representation of $\dot{\Delta}_{\pm}^{(0,\nu)}(p,q)$.

Example: Assume that $B>0$, $p>0$, and $\Theta=-\pi/2$. Then, setting in (4.48) $y=p/B$, $y'=q/B$, and taking into account (4.49), one finds a triple integral representation for $\dot{\Delta}_{+}^{(0,\nu)}(p,q)$

$$\dot{\Delta}_{+}^{(0,\nu)}(p,q)=\sqrt{2B}\frac{pq}{B^2}\frac{2^{1+\lambda^2/2B}\left[\Gamma\left(\frac{\lambda^2}{4B}+1\right)\right]^2}{(2\pi)^{5/2}}\frac{\sin \pi\nu}{\pi}\int_{-\infty}^{\infty}d\theta\int_0^{\pi}d\varphi\int_0^{\pi}d\varphi'\frac{e^{(1+\nu)(\theta+i(\varphi-\varphi'))}}{e^{\theta+i(\varphi-\varphi')}+1}$$
$$\times\frac{1}{\sin^2\varphi\sin^2\varphi'}\exp\left\{-\frac{1}{4B}\left(\frac{p^2}{\sin^2\varphi}+\frac{q^2}{\sin^2\varphi'}+\frac{2pqe^{\theta}}{\sin\varphi\sin\varphi'}+2ip^2\operatorname{ctg}\varphi\right.\right.$$
$$\left.\left.-2iq^2\operatorname{ctg}\varphi'\right)\right\}\times U\left(\frac{\lambda^2}{2B}+1,1,\frac{1}{2B}\left(\frac{p^2}{\sin^2\varphi}+\frac{q^2}{\sin^2\varphi'}+\frac{2pq\cosh\theta}{\sin\varphi\sin\varphi'}\right)\right).$$

Much simpler results can be obtained for $B=0$. In this case, it is convenient to introduce instead of the momentum p a rapidity variable θ_p defined by

$$p=\lambda\sinh\theta_p,\qquad \sqrt{\lambda^2+p^2}=\lambda\cosh\theta_p.$$

Partial waves (4.34) can then be written as

$$\Phi^{(\pm)}(p,y)=\frac{e^{\mp\lambda y\cosh\theta_p}}{\sqrt{2\cosh\theta_p}}\begin{pmatrix} C_+^{-1}e^{\pm\theta_p/2}\\ \pm iC_+e^{\mp\theta_p/2}\end{pmatrix}.$$

Set $\Theta=-\pi/2$ and substitute the representation (4.29) for $\Delta(z,z')$ (recall that it is valid for $y,y'>0$) into (4.48) and (4.49). After elementary integration over x and x' in the left hand side of (4.48) we find

$$\dot{\Delta}_{\pm}^{(0,\nu)}(p,q)=\frac{1}{\lambda\sqrt{\cosh\theta_p\cosh\theta_q}}\frac{\sin\pi\nu}{\pi}\frac{e^{\mp(1+\nu)(\theta_p+\theta_q)}}{2\cosh\frac{\theta_p+\theta_q}{2}}. \tag{4.50}$$

Similarly, for $\Theta=\pi/2$ one obtains

$$\dot{\Delta}_{\pm}^{(0,\nu)}(p,q)=-\frac{1}{\lambda\sqrt{\cosh\theta_p\cosh\theta_q}}\frac{\sin\pi\nu}{\pi}\frac{e^{\mp\nu(\theta_p+\theta_q)}}{2\cosh\frac{\theta_p+\theta_q}{2}}. \tag{4.51}$$

It should be mentioned that the formulas equivalent to (4.50) and (4.51) were first obtained by Schroer and Truong.[27] In a context similar to ours, they were rediscovered by Palmer in Ref. 4.

We finally comment on the limiting form of Eq. (1.1) in flat space. Introduce $s=R^{-2}t$ and let $R\to\infty$. Then, setting $w(s)=1-y(t)$ and using asymptotic behavior of the parameters β and δ in (1.3)

$$\beta-\delta\simeq R^2\gamma',\qquad \delta\simeq R^4\delta',$$

$$\gamma'=\frac{m^2-E^2+B(1+\nu_1+\nu_2)}{2},\qquad \delta'=-\frac{B^2}{8}, \tag{4.52}$$

it is straightforward to check that $y(t)$ satisfies Painlevé V equation

$$\frac{d^2y}{dt^2}=\left(\frac{1}{2y}+\frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2-\frac{1}{t}\frac{dy}{dt}+\frac{(y-1)^2}{t^2}\left(\alpha'y+\frac{\beta'}{y}\right)+\frac{\gamma'y}{t}+\frac{\delta'y(y+1)}{y-1} \tag{4.53}$$

with parameters $\alpha'=\alpha=\lambda^2/2$, $\beta'=0$ and γ', δ' defined by (4.52). Equation (1.2) in the planar limit transforms into

$$t\frac{d}{dt}\ln\,\tau(t) = \frac{t^2}{4y(y-1)^2}\left(\frac{dy}{dt}\right)^2 - \frac{\lambda^2 y}{4} + \frac{\eta\theta}{2}\frac{ty}{y-1} - \frac{\eta^2}{4}\frac{t^2 y}{(y-1)^2}, \tag{4.54}$$

where $\eta=-B/2$ and $\eta(\theta+1)=\gamma'$. The right hand side of (4.54) coincides, up to addition of a constant, with the Okamoto Hamiltonian for Painlevé V Eq. (4.53) and thus the τ-function (4.46) associated to the Dirac operator in flat space is very simply related to Painlevé V τ-function. Although one could expect a similar relation in the case of Painlevé VI and Dirac operator on the hyperbolic disk, this appears not to be the case.[14]

V. CONCLUDING REMARKS

An important problem left outside the scope of this paper concerns the short-distance ($s\to 0$) behavior of the PBT τ-function. Extending the conjecture of Doyon[17] to the case $b\neq 0$, one could assume that $\tau(s)$ coincides with the two-point correlator of twist fields in the Dirac theory in a more general classical background (Poincaré metric+uniform magnetic field). External fields drastically change the infrared asymptotics of the correlation function, but they should not affect the exponent σ in its conformal behavior $\tau(s\to 0)\simeq Cs^{\sigma}$. To prove this rigorously, one would require a generalization of the connection formulas for PVI (Refs. 28 and 29) to nongeneric values of parameters (recall that in our case $\gamma=0$). These formulas, however, are not known except for the special case of PVI with $\beta=\gamma=0$, $\delta=\frac{1}{2}$.[30,31]

The theory of Painlevé equations does not provide any answer for the coefficient C. For the τ-function arising in the scaling limit of the 2D Ising model, this constant was extracted from a careful asymptotic analysis of the corresponding Fredholm determinant.[32] Although it seems hopeless to repeat such an analysis with the determinant (1.4), one could try to obtain C using QFT arguments: it may be expressed in terms of the vacuum expectation values of twist fields, which can be computed by the method of angular quantization.[17,33]

It is curious to note that another one-parameter class of solutions of the PVI equation with one singular parameter arises in the representation theory of the infinite-dimensional unitary group.[34] However, its relation to PVI transcendents studied in the present paper remains rather obscure.

APPENDIX: INVARIANCE OF THE τ-FUNCTION

We wish to show that the logarithmic derivative of the τ-function is invariant under the joint $SU(1,1)$-transformation of the branch point positions:

$$a_j \mapsto \frac{\alpha a_j+\beta}{\bar{\beta}a_j+\bar{\alpha}}, \quad \begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix} \in SU(1,1).$$

It is sufficient to prove that the one-form (3.88) is annihilated by two vector fields

$$X = \sum_j (a_j\partial_{a_j} - \bar{a}_j\partial_{\bar{a}_j}),$$

$$Y = \sum_j [(1-a_j^2)\partial_{a_j} + (1-\bar{a}_j^2)\partial_{\bar{a}_j}],$$

whose integral curves are orbits of a compact and a noncompact subgroup of $SU(1,1)$.

Let us first apply X to $d\ln\tau$. Substituting (3.83) and (3.84) into the resulting expression, one finds after some simplifications

$$X(d\ln\tau) = \sum_j\sum_{k\neq j}\frac{1}{1-|a_j|^2}(a_j e_{jk}a^j_{1/2,k} + \bar{a}_j h_{jk}d^j_{1/2,k}) + \sum_{j,k}\frac{1}{1-|a_j|^2}(a_j\bar{a}_k f_{jk}c^j_{1/2,k} + \bar{a}_j a_k g_{jk}b^j_{1/2,k}). \tag{A1}$$

Two sums in (A1) are separately equal to zero. For example, the first one can be rewritten as

$$\sum_j \sum_{k\neq j} \{e_{jk}(\mathbf{a}_1 A)_{kj} + h_{jk}(\mathbf{d}_1\bar{A})_{kj}\} = \sum_{j,k} \{[\mathbf{a}_1,A]_{jk}(\mathbf{a}_1 A)_{kj} - [\mathbf{d}_1,\bar{A}]_{jk}(\mathbf{d}_1\bar{A})_{kj}\}$$
$$= \frac{1}{2}\mathrm{Tr}\{[\mathbf{a}_1,A]^2 - [\mathbf{d}_1,\bar{A}]^2\}$$
$$= \frac{1}{2}\mathrm{Tr}\{(e-\Lambda+b\mathbf{1})^2 - (h-\Lambda+b\mathbf{1})^2\}$$
$$= \frac{1}{2}\mathrm{Tr}\{fg - gf - 2(\Lambda - b\mathbf{1})(e-h)\} = 0.$$

Besides the relations (3.73) and (3.79), in the above we have used the fact that the diagonal parts of the commutators $[\mathbf{a}_1, A]$ and $[\mathbf{d}_1, \bar{A}]$ and of the difference $e-h$ are equal to zero. Similarly, the second sum in (A1) gives

$$\mathrm{Tr}(f\bar{A}\mathbf{c}_1 A + gA\mathbf{b}_1\bar{A}) = \mathrm{Tr}((A\mathbf{b}_1\bar{A} - \mathbf{b}_1)\bar{A}\mathbf{c}_1 A + (\mathbf{c}_1 - \bar{A}\mathbf{c}_1 A)A\mathbf{b}_1\bar{A}) = 0.$$

Next consider the action of Y. We get

$$Y(d\ln\tau) = \sum_j \sum_{k\neq j} \{e_{jk}(\mathbf{a}_1(1-A^2))_{kj} - h_{jk}(\mathbf{d}_1(1-\bar{A}^2))_{kj}\} + \sum_j (a_j+\bar{a}_j)m_+(\tilde{\nu}_j - 1/2, b) + \sum_{j,k}\{f_{jk}(\bar{A}\mathbf{c}_1 - \bar{A}\mathbf{c}_1 A^2)_{kj} - g_{jk}(A\mathbf{b}_1 - A\mathbf{b}_1\bar{A}^2)_{kj}\}. \tag{A2}$$

The first sum in (A2) can be transformed into

$$\sum_j \sum_{k\neq j} \{-e_{jk}(\mathbf{a}_1 A^2)_{kj} + h_{jk}(\mathbf{d}_1\bar{A}^2)_{kj}\} = -\mathrm{Tr}([\mathbf{a}_1,A]^2 A + [\mathbf{d}_1,\bar{A}]^2\bar{A})$$
$$= -\mathrm{Tr}(e^2 A + h^2\bar{A} - 2(\Lambda - b\mathbf{1})(eA + h\bar{A}) + (\Lambda - b\mathbf{1})^2(A+\bar{A}))$$
$$= -\mathrm{Tr}(e^2 A + h^2\bar{A} - (\Lambda - b\mathbf{1})^2(A+\bar{A})),$$

while the third one gives

$$\mathrm{Tr}(f(\bar{A}\mathbf{c}_1 - \mathbf{c}_1 A + gA) - g(A\mathbf{b}_1 - \mathbf{b}_1\bar{A} - f\bar{A})) = \mathrm{Tr}(fgA + gf\bar{A}).$$

Summing up the three contributions in (A2) and using (3.79) once again, one finds that $d\ln\tau$ is invariant under the flow of Y.

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