Finite-volume correlation functions of monodromy fields on the lattice: Toeplitz representation

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Abstract

A family of interacting local fields, generalizing disorder variables of the 2D Ising model, is constructed from free Dirac fermions on the lattice. We express vacuum expectation values of these fields and form factor expansions of their correlation functions in terms of determinants and inverses of Toeplitz matrices of remarkably simple form.

1 Introduction

In recent years, an increasing attention has been paid to the study of correlation functions in quantum field theory (QFT) in a finite volume. Matsubara imaginary time formalism relates such correlation functions to the correlation functions at non-zero temperature, which are the quantities of principal interest in condensed matter theory.

It is common to represent correlation functions in massive theories via Callen-Lehmann expansion over intermediate eigenstates. Building blocks for this representation are eigenvalues of the hamiltonian and form factors, i. e. matrix elements of the field operators in the basis of corresponding eigenstates. The main achievements in the investigation of finite-volume spectrum are related to integrable 2D QFTs. Two methods of calculation of form factors in integrable QFT, namely, bootstrap approach [19] and angular quantization [2], do not work in the finite volume. Therefore, only a few results are known so far, in particular, exact vacuum expectation values in the sinh-Gordon model [15] and spin form factors in the Ising field theory [3, 4, 9].

A possible way to obtain exact finite-volume form factors and correlation functions for other integrable QFTs is following. One should find first an integrable lattice regularization of the theory, then to compute transfer matrix spectrum and form factors on the finite lattice and then to consider the appropriate limit. Exactly this scheme has been realized for the Ising model. Another case where it hopefully works is the sine-Gordon field theory and related to it massive Thirring model: it is known that the latter system arise in the scaling limit of an inhomogeneous six-vertex model [6], which is in turn related to XXZ quantum spin chain. Form factors of the last system have been recently calculated in [12].

The present paper is devoted to the calculation of the finite-volume vacuum expectation values and two-point correlation functions of lattice monodromy fields [5, 16]. These fields are lattice analogs of the exponential fields of the sine-Gordon model at the free-fermion point. Infinite-volume form factors of the exponential fields can be calculated in several ways [1, 17, 18], but none of the existing methods can be generalized to the finite volume (see, however, [7, 8]). Besides the lattice approach, developed in this work, one may also hope to obtain finite-volume correlators of the exponential fields directly in the continuum limit, using monodromy preserving deformation theory for the Dirac equation on the cylinder [13]. Actually, we have already obtained the simplest one-particle form factors in this way.

This paper is organized as follows. In Section 2, monodromy fields are constructed from free Dirac fermions on the lattice. Any finite-volume correlation function of these fields can be formally written as the determinant of the Dirac operator on a cylindrical lattice with defects. Sections 3 and 4 are devoted to vacuum expectation values and two-point correlation functions of monodromy fields. These quantities are expressed in terms of determinants and inverses of certain Toeplitz matrices of size *independent* of the separation of correlating fields. The formulae for correlation functions have the form of form factor expansions. In Section 5, after a simple check of our results, we present explicit expressions for the determinants of relevant Toeplitz matrices in one nontrivial case: for a particular choice of monodromy lattice exponential fields are related to the disorder variables of the two-dimensional Ising model. This is a lattice version of the well-known connection between the Ising field theory and sine-Gordon model at the free-fermion point [18].

2 Lattice Dirac operator and monodromy fields

We wish to consider the lattice drawn in the Fig. 1. To each site of the lattice we attach a two-component complex fermion field $\psi(r_x, r_y)$, where $r_x \in \mathbb{Z}$, $r_y = 0, 1, \ldots, N-1$ and quasiperiodic boundary conditions are imposed:

$$\psi(r_x, r_y + N) = e^{2\pi i\alpha}\psi(r_x, r_y).$$

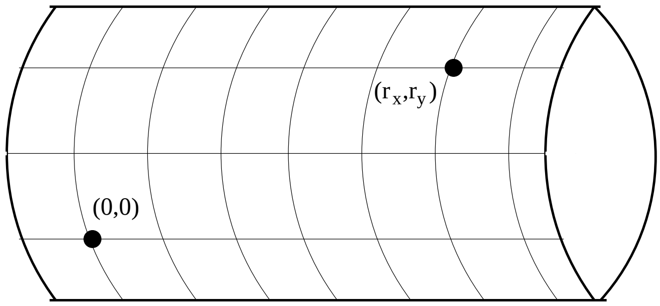


Fig. 1

Let us introduce lattice shifts ∇_x, ∇_y

$$\nabla_x \psi(r_x, r_y) = \psi(r_x+1, r_y), \qquad \nabla_y \psi(r_x, r_y) = \psi(r_x, r_y+1)$$

and define lattice Dirac operator as follows

$$\hat{D} = \begin{pmatrix} 1 - t\nabla_y & 1 - t\nabla_x \\ -(1 - t\nabla_{-x}) & 1 - t\nabla_{-y} \end{pmatrix}. \tag{1}$$

This choice of regularization is prompted by the study of the two-dimensional Ising model. Parameter t, which is assumed to be real and positive, is related to the Ising coupling constant $\mathcal{K}$ by $t = \sinh 2\mathcal{K}$.

Recall that in the definition of the usual continuous Dirac operator

$$D = m + \gamma_x \partial_x + \gamma_y \partial_y$$

one can suppose that γ_x, γ_y are given by two arbitrary Pauli matrices. To recover this continuous operator, one should first consider the following “naive” continuum limit of (1):

$$\nabla_x \to 1 + a\partial_x, \qquad \nabla_y \to 1 + a\partial_y,$$

$$t - 1 \to -\frac{ma}{\sqrt{2}}, \qquad \psi \to \sqrt{a}\,\psi, \qquad a \to 0.$$

Operator (1) transforms then into

$$D_{naive} \to \frac{m}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} + \sigma_x \partial_x + \sigma_z \partial_y,$$

where we denote as usual

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In order to make the mass term diagonal, one can apply the transformation $\psi \to e^{i\sigma_y \pi/8}\psi$. Since for Dirac conjugate we have

$$\bar{\psi} = \psi^\dagger \sigma_z \to \psi^\dagger e^{-i\sigma_y \pi/8}\sigma_x = \psi^\dagger \sigma_x e^{i\sigma_y \pi/8} = \bar{\psi}\, e^{i\sigma_y \pi/8},$$

the operator D_{naive} transforms into

$$D = m + \sigma_x \partial_x + \sigma_z \partial_y.$$

This continuum limit can be also performed more accurately, i.e. at the quantum level, but we will not pursue this question further.

In two dimensions, there is an interesting way to define new local fields which are manifestly nonlocal in terms of old ones. The most known example is provided by the correspondence between the sine-Gordon model and massive Thirring fermions. Sine-Gordon bosons are nonlocal in terms of fermion fields, which yields their nontrivial scattering even at the free-fermion point.

The same phenomenon on the lattice happens, for instance, in the two-dimensional Ising model, which is related to free Majorana fermions. The local free fermion field here is nonlocal in terms of spin variables (Wigner strings). It is possible to construct yet another local field, called disorder variable. In terms of spin variables, this latter represents a magnetic dislocation which starts at a given point and goes to infinity.

In general, such a phenomenon is possible when the theory has a continuous or discrete symmetry. We will illustrate the general idea on the example of complex free fermions on the lattice. These fermions are characterized by the action

$$S[\psi, \bar{\psi}] = \sum_{r_x, r_y} \bar{\psi}(r_x, r_y) \hat{D}\, \psi(r_x, r_y).$$

Now consider a closed path P on the dual lattice, shown in the Fig. 2 by dashed lines (it can also have self-intersections). Let us make the change of variables $\psi \to e^{2\pi i \nu}\psi$ at all sites enclosed by this contour. Due to global $U(1)$-invariance, the action will change only on the edges intersected by P.

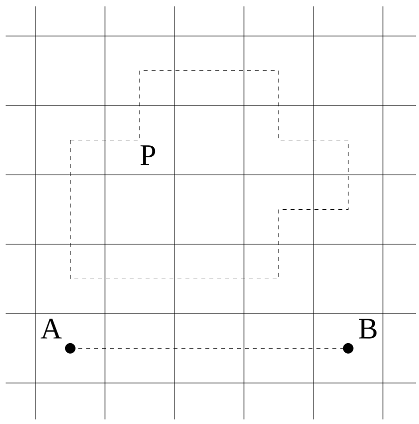


Fig. 2

The calculation of correlation functions includes the integration over all Grassmann fields. It is easy to understand that any described closed defect of the action is equivalent to zero under the functional integral, since it can be removed by a change of variables. Therefore, if we add to the action a term corresponding to some open defect, the integral will depend on its endpoints A and B, but not on the concrete shape of the path joining them. If the points A and B are located on the line, parallel to the cylinder axis, then this additional term can be taken in the following form:

$$\delta S_{AB} = t \sum_{r_x=0}^{r-1} \Big(\xi \bar{\psi}^1(r_x, N-1)\psi^1(r_x, 0) + \xi^* \bar{\psi}^2(r_x, 0)\psi^2(r_x, N-1) \Big),$$

$$\xi = e^{2\pi i \alpha}(1 - e^{2\pi i \nu}).$$

It can also be encoded into the change of boundary conditions:

$$\begin{cases} \psi(r_x, r_y + N) = e^{2\pi i \alpha}\psi(r_x, r_y) & \text{for } r_x < 0 \text{ and } r_x > r-1, \\ \psi(r_x, r_y + N) = e^{2\pi i (\alpha+\nu)}\psi(r_x, r_y) & \text{for } 0 \leq r_x \leq r-1. \end{cases}$$

Now one can define two-point correlation function of monodromy fields as a normalized partition function of the Dirac model with the defect. It can be written as the ratio of determinants of the corresponding lattice Dirac operators:

$$\langle \mathcal{O}_{\alpha,\alpha+\nu}(A)\mathcal{O}_{\alpha+\nu,\alpha}(B)\rangle = \frac{\int d[\psi,\bar{\psi}]\ e^{S[\psi,\bar{\psi}]+\delta S_{AB}}}{\int d[\psi,\bar{\psi}]\ e^{S[\psi,\bar{\psi}]}} = \frac{\det \hat{D}^{\,def}}{\det \hat{D}}.$$

Generalization to the multipoint case is straightforward. Monodromy fields live on the dual lattice and can be thought of as open defects that start from a given point and go to infinity. Formally one can write

$$\mathcal{O}_{\alpha+\nu,\alpha}\left(x+\frac{1}{2},y+\frac{1}{2}\right) =$$
$$\exp\left\{t\sum_{r_x=-\infty}^{x}\Big(\xi\bar{\psi}^1(r_x,y)\psi^1(r_x,y+1)+\xi^*\bar{\psi}^2(r_x,y+1)\psi^2(r_x,y)\Big)\right\}.$$

Locality of these fields implies, in particular, that their correlation functions can be represented via form factor expansions.

Consider, for instance, two-point correlator $\langle\mathcal{O}_{\alpha,\alpha+\nu}(0,0)\mathcal{O}_{\alpha+\nu,\alpha}(r_x,r_y)\rangle$. In this case it is convenient to divide the axis of discrete time into three intervals: $(-\infty;0)$, $[0,r_x)$ and $[r_x;\infty)$. The evolution is governed by the hamiltonian of free Dirac fermions on the lattice with different periodicity conditions in each interval. Taking this into account, one obtains

$$\langle\mathcal{O}_{\alpha,\alpha+\nu}(0,0)\mathcal{O}_{\alpha+\nu,\alpha}(r_x,r_y)\rangle =$$
$$\sum_n {}_{\alpha}\langle vac|\mathcal{O}_{\alpha,\alpha+\nu}(0,0)|n\rangle_{\alpha+\nu}\ {}_{\alpha+\nu}\langle n|\mathcal{O}_{\alpha+\nu,\alpha}(0,0)|vac\rangle_{\alpha}\ e^{-r_xE_n+ir_yP_n},$$

where n labels orthonormal multiparticle eigenstates of the hamiltonian, and E_n, P_n denote their energies and quasimomenta. This is the form of the answer that we expect to obtain.

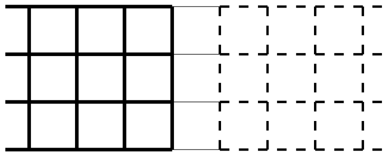
3 Vacuum expectation values of monodromy fields

First term of the infrared asymptotics of correlation functions is determined by the product of vacuum expectation values

$$\mathfrak{M}^2_{\alpha,\alpha+\nu} = {}_{\alpha}\langle vac|\mathcal{O}_{\alpha,\alpha+\nu}(0,0)|vac\rangle_{\alpha+\nu}\ {}_{\alpha+\nu}\langle vac|\mathcal{O}_{\alpha+\nu,\alpha}(0,0)|vac\rangle_{\alpha} \qquad (2)$$

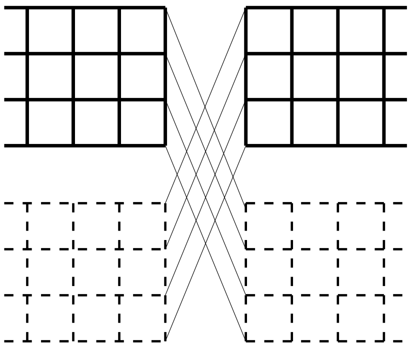
Note that since our field operator intertwines Hilbert spaces which correspond to different periodicity conditions, the vacua on its left and right hand side are different vectors.

Monodromy fields are well-defined only inside correlation functions. In order to calculate their matrix elements, one should first regularize them.
The problem is basically the following. Vacuum expectation value ${}_{\alpha}\langle vac|\mathcal{O}_{\alpha,\alpha+\nu}(0,0)|vac\rangle_{\alpha+\nu}$ is proportional to the (infinite) partition function of free fermions, which obey different boundary conditions for $r_x>0$ and $r_x<0$. We will correspond to this partition function the following picture:

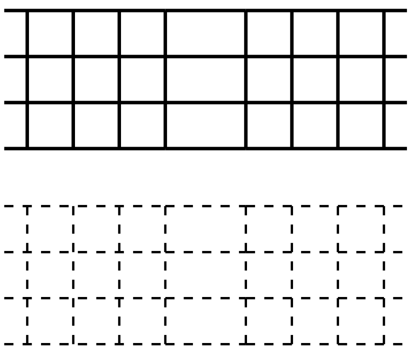


How can one find the appropriate proportionality factor without calculating two-point correlation function first?

The idea is to consider the product (2), which is proportional to



If we formally divide this by the product of two partition functions without the defects but with different boundary conditions for fermions, symbolically represented by



then the result will be finite and meaningful. It will be shown later that such a ratio gives us the product (2). It coincides with the normalized partition function of the system of two free fermion fields ψ, φ with a circular defect:

$$\mathfrak{M}^2_{\alpha,\alpha+\nu} = \frac{\int d[\psi,\bar{\psi}]\, d[\varphi,\bar{\varphi}]\; e^{\,\bar{\psi}\hat{D}^{(\alpha)}\psi+\bar{\varphi}\hat{D}^{(\nu+\alpha)}\varphi+\delta S_1}}{\int d[\psi,\bar{\psi}]\, d[\varphi,\bar{\varphi}]\; e^{\,\bar{\psi}\hat{D}^{(\alpha)}\psi+\bar{\varphi}\hat{D}^{(\nu+\alpha)}\varphi}}, \tag{3}$$

where the explicit form of δS_1 is given by

$$\begin{aligned}
\delta S_1 &= t\sum_{r_y=0}^{N-1}\Big\{\bar\psi^1(0,r_y)\psi^2(1,r_y)-\bar\psi^2(1,r_y)\psi^1(0,r_y)+\bar\varphi^1(0,r_y)\varphi^2(1,r_y)-\\
&\quad -\bar\varphi^2(1,r_y)\varphi^1(0,r_y)-\bar\psi^1(0,r_y)\varphi^2(1,r_y)+\bar\varphi^2(1,r_y)\psi^1(0,r_y)-\\
&\quad -\bar\varphi^1(0,r_y)\psi^2(1,r_y)+\bar\psi^2(1,r_y)\varphi^1(0,r_y)\Big\}\\
&= t\sum_{r_y=0}^{N-1}\Big\{\big(\bar\varphi^1(0,r_y)-\bar\psi^1(0,r_y)\big)\big(\varphi^2(1,r_y)-\psi^2(1,r_y)\big)-\\
&\quad -\big(\bar\varphi^2(1,r_y)-\bar\psi^2(1,r_y)\big)\big(\varphi^1(0,r_y)-\psi^1(0,r_y)\big)\Big\}.
\end{aligned}$$

It is convenient to represent the extra factor $e^{\,\delta S_1}$ as an integral over auxiliary two-component complex Grassmann field μ, living on the "circle" of length N:

$$e^{\,\delta S_1}=e^{\bar\chi P^T J P\chi}=\int d[\mu,\bar\mu]\,e^{\,\bar\mu J\mu+\bar\mu P\chi-\bar\chi P^T\mu},$$

where

$$J=\begin{pmatrix}0 & \mathbf{1}_N\\ -\mathbf{1}_N & 0\end{pmatrix},\qquad \chi=\begin{pmatrix}\varphi^1-\psi^1\\ \varphi^2-\psi^2\end{pmatrix},\tag{4}$$

$$P=\sqrt{t}\begin{pmatrix}\mathbf{1}_N\,\delta_{0,r_x} & 0\\ 0 & \mathbf{1}_N\,\delta_{1,r_x}\end{pmatrix}.\tag{5}$$

Changing the order of integration in the numerator of (3) and using well-known formulae, one can easily evaluate the integrals over initial fields ψ and φ. The integral over auxiliary field then gives

$$\mathfrak{M}^2_{\alpha,\alpha+\nu}=\int d[\mu,\bar\mu]\,e^{\,-\bar\mu G\mu}=\det G.$$

Here, the matrix G is given by

$$G=\begin{pmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{pmatrix}=-J+P\left(\hat D^{(\alpha)}\right)^{-1}P^T+P\left(\hat D^{(\alpha+\nu)}\right)^{-1}P^T.\tag{6}$$

Its $N\times N$ blocks G_{ij} $(i,j=1,2)$ can be written more explicitly. They may be expressed in terms of the inverse matrix elements of Dirac operators corresponding to different boundary conditions:

$$(G_{11})_{yy'}=t\left[\left(\hat D^{(\alpha)}\right)^{-1}_{11}\right]_{(0y)(0y')}+t\left[\left(\hat D^{(\alpha+\nu)}\right)^{-1}_{11}\right]_{(0y)(0y')},\tag{7}$$

$$(G_{12})_{yy'} = -\delta_{yy'} + t\left[\left(\hat{D}^{(\alpha)}\right)^{-1}_{12}\right]_{(0y)(1y')} + t\left[\left(\hat{D}^{(\alpha+\nu)}\right)^{-1}_{12}\right]_{(0y)(1y')}, \tag{8}$$

$$(G_{21})_{yy'} = \delta_{yy'} + t\left[\left(\hat{D}^{(\alpha)}\right)^{-1}_{21}\right]_{(1y)(0y')} + t\left[\left(\hat{D}^{(\alpha+\nu)}\right)^{-1}_{21}\right]_{(1y)(0y')}, \tag{9}$$

$$(G_{22})_{yy'} = t\left[\left(\hat{D}^{(\alpha)}\right)^{-1}_{22}\right]_{(1y)(1y')} + t\left[\left(\hat{D}^{(\alpha+\nu)}\right)^{-1}_{22}\right]_{(1y)(1y')}. \tag{10}$$

The Dirac operator and its inverse are given by diagonal matrices in the Fourier representation. Using the inverse transformation, it is then straightforward to obtain

$$\left(\hat{D}^{(\alpha)}\right)^{-1}_{(xy)(x'y')} = \frac{1}{2\pi N}\sum_{p_y}^{(\alpha)}\int_{-\pi}^{\pi} dp_x \begin{pmatrix} 1-te^{-ip_y} & -1+te^{ip_x} \\ 1-te^{-ip_x} & 1-te^{ip_y} \end{pmatrix} \frac{e^{ip_x(x-x')+ip_y(y-y')}}{2t\, D_{p_xp_y}}. \tag{11}$$

Here the sum $\sum_{p_y}^{(\alpha)}$ is performed over the discrete values of the second component of quasimomentum: $p_y = \frac{2\pi}{N}(j+\alpha)$ $(j = 0, 1, \ldots, N-1)$. We have also introduced the notation

$$D_{p_xp_y} = t + t^{-1} - \cos p_x - \cos p_y.$$

Substituting (11) into the formulae (7)–(10) for matrix elements of G, one finds

$$\begin{aligned}(G_{11})_{yy'} &= \frac{1}{2\pi N}\sum_{p_y}^{(\alpha)}\int_{-\pi}^{\pi} dp_x\, \frac{1-te^{-ip_y}}{2D_{p_xp_y}}\, e^{ip_y(y-y')} + \\ &+ \frac{1}{2\pi N}\sum_{p_y}^{(\alpha+\nu)}\int_{-\pi}^{\pi} dp_x\, \frac{1-te^{-ip_y}}{2D_{p_xp_y}}\, e^{ip_y(y-y')},\end{aligned}$$

$$\begin{aligned}(G_{22})_{yy'} &= \frac{1}{2\pi N}\sum_{p_y}^{(\alpha)}\int_{-\pi}^{\pi} dp_x\, \frac{1-te^{ip_y}}{2D_{p_xp_y}}\, e^{ip_y(y-y')} + \\ &+ \frac{1}{2\pi N}\sum_{p_y}^{(\alpha+\nu)}\int_{-\pi}^{\pi} dp_x\, \frac{1-te^{ip_y}}{2D_{p_xp_y}}\, e^{ip_y(y-y')},\end{aligned}$$

$$(G_{12})_{yy'} = -\,(G_{21})_{yy'} = -\frac{1}{2t}\left[(G_{11})_{yy'} + (G_{22})_{yy'}\right].$$

The expressions defining G can be simplified even more. Consider the matrix

$$\tilde{G} = R\,G\,R, \qquad R = e^{\,i\sigma_y\theta/2}, \qquad \operatorname{ctg}\theta = t.$$

One may verify that $\tilde{G}$ is block-diagonal:

$$\left(\tilde{G}_{12}\right)_{yy'} = \left(\tilde{G}_{21}\right)_{yy'} = 0,$$

$$\left(\tilde{G}_{11}\right)_{yy'} = \frac{\sqrt{1+t^2}+t}{2t}\,(G_{11})_{yy'} + \frac{\sqrt{1+t^2}-t}{2t}\,(G_{22})_{yy'}\,,$$

$$\left(\tilde{G}_{22}\right)_{yy'} = \frac{\sqrt{1+t^2}-t}{2t}\,(G_{11})_{yy'} + \frac{\sqrt{1+t^2}+t}{2t}\,(G_{22})_{yy'}\,.$$

The integrals over continuous quasimomentum p_x can be calculated by residues. We obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{dp_x}{D_{p_x p_y}} = \frac{1}{\sinh\gamma(p_y)},$$

where the function $\gamma(p) > 0$ is given by the positive root of the equation

$$\cosh\gamma(p) = t + t^{-1} - \cos p,$$

which represents a lattice analog of the relativistic energy dispersion low. Note that the following relations hold:

$$\frac{\sqrt{1+t^2}\pm 1}{t} = \begin{cases} e^{\pm\frac{\gamma(\pi)+\gamma(0)}{2}} & \text{for } 0 < t < 1, \\ e^{\pm\frac{\gamma(\pi)-\gamma(0)}{2}} & \text{for } t > 1, \end{cases}$$

$$\sqrt{1+t^2}\pm t = \begin{cases} e^{\pm\frac{\gamma(\pi)-\gamma(0)}{2}} & \text{for } 0 < t < 1, \\ e^{\pm\frac{\gamma(\pi)+\gamma(0)}{2}} & \text{for } t > 1. \end{cases}$$

Taking into account the above remarks, it is straightforward to obtain the representation for the product (2) of the vacuum expectation values of monodromy fields in terms of $N \times N$ Toeplitz matrices:

$$\mathfrak{M}^2_{\alpha,\alpha+\nu} = \det\tilde{G}_{11}\,\det\tilde{G}_{22}$$

$$= \det\left(\frac{A^{(\alpha)} + A^{(\alpha+\nu)}}{2}\right)\,\det\left(\frac{\left(A^{(\alpha)}\right)^{-1} + \left(A^{(\alpha+\nu)}\right)^{-1}}{2}\right), \qquad (12)$$

$$A_{yy'}^{(\alpha)} = \frac{1}{N} \sum_{p_y}^{(\alpha)} C\left(e^{ip_y}\right) e^{ip_y(y-y')}, \qquad y, y' = 0, 1, \ldots, N-1. \tag{13}$$

The form of the function $C(z)$ is different in the "ferromagnetic" ($t > 1$) and "paramagnetic" ($0 < t < 1$) region of values of the parameter t, and is given by the known Toeplitz kernels arising in the study of the two-dimensional Ising model:

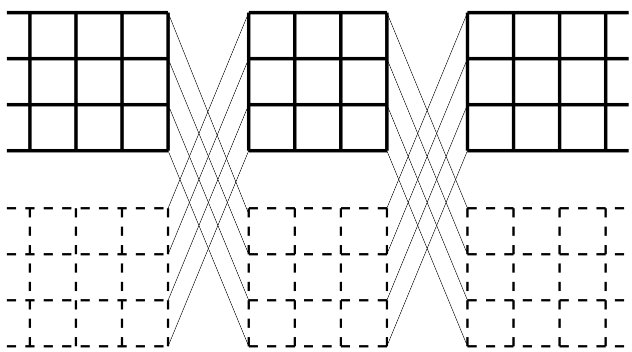
$$C(z) = \begin{cases} \sqrt{\dfrac{\left(1-e^{-\gamma(0)}z^{-1}\right)\left(1-e^{-\gamma(\pi)}z\right)}{\left(1-e^{-\gamma(0)}z\right)\left(1-e^{-\gamma(\pi)}z^{-1}\right)}} & \text{for } 0 < t < 1, \\ -z^{-1}\sqrt{\dfrac{\left(1-e^{-\gamma(0)}z\right)\left(1-e^{-\gamma(\pi)}z\right)}{\left(1-e^{-\gamma(0)}z^{-1}\right)\left(1-e^{-\gamma(\pi)}z^{-1}\right)}} & \text{for } t > 1. \end{cases} \tag{14}$$

4 Correlation functions of monodromy fields

The calculation of correlation functions is based on the same technical ideas as the previous construction. Namely, we will study the product

$$\mathfrak{F}_{\alpha,\alpha+\nu}(r) = \langle \mathcal{O}_{\alpha,\alpha+\nu}(0,0)\mathcal{O}_{\alpha+\nu,\alpha}(r,0)\rangle \, \langle \mathcal{O}_{\alpha+\nu,\alpha}(0,0)\mathcal{O}_{\alpha,\alpha+\nu}(r,0)\rangle, \tag{15}$$

which can be represented by the following picture (the normalization is understood):



Now we have two circular defects. The additional term δS_2 in the expression for the product of correlation functions

$$\mathfrak{F}_{\alpha,\alpha+\nu}(r) = \frac{\int d[\psi,\bar\psi]\, d[\varphi,\bar\varphi]\; e^{\,\bar\psi \hat D^{(\alpha)}\psi + \bar\varphi \hat D^{(\nu+\alpha)}\varphi + \delta S_2}}{\int d[\psi,\bar\psi]\, d[\varphi,\bar\varphi]\; e^{\,\bar\psi \hat D^{(\alpha)}\psi + \bar\varphi \hat D^{(\nu+\alpha)}\varphi}}$$

is given by

$$\begin{aligned}\delta S_2 &= t\sum_{r_y=0}^{N-1}\Big\{\big(\bar\varphi^1(0,r_y)-\bar\psi^1(0,r_y)\big)\big(\varphi^2(1,r_y)-\psi^2(1,r_y)\big)-\\ &\quad-\big(\bar\varphi^2(1,r_y)-\bar\psi^2(1,r_y)\big)\big(\varphi^1(0,r_y)-\psi^1(0,r_y)\big)+\\ &\quad+\big(\bar\varphi^1(r,r_y)-\bar\psi^1(r,r_y)\big)\big(\varphi^2(r+1,r_y)-\psi^2(r+1,r_y)\big)-\\ &\quad-\big(\bar\varphi^2(r+1,r_y)-\bar\psi^2(r+1,r_y)\big)\big(\varphi^1(r,r_y)-\psi^1(r,r_y)\big)\Big\}\\ &= \bar\chi P^T J P\chi+\bar\chi Q^T J Q\chi,\end{aligned}$$

where J, P, χ are defined by formulae (4)–(5) and

$$Q=\sqrt{t}\begin{pmatrix}\mathbf{1}_N\,\delta_{r,r_x} & 0\\ 0 & \mathbf{1}_N\,\delta_{r+1,r_x}\end{pmatrix}.$$

Again, it is convenient to represent the extra factor $e^{\delta S_2}$ as an integral, this time over two-auxiliary two-component complex fermion fields μ, η:

$$e^{\delta S_2}=\int d[\mu,\bar\mu]\,d[\eta,\bar\eta]\,e^{\,\bar\mu J\mu+\bar\mu P\chi-\bar\chi P^T\mu+\bar\eta J\eta+\bar\eta Q\chi-\bar\chi Q^T\eta}.$$

After the change of order of integration we obtain

$$\mathfrak{F}_{\alpha,\alpha+\nu}(r)=\int d[\mu,\bar\mu]\,d[\eta,\bar\eta]\,e^{\,-(\bar\mu\;\bar\eta)\,H\,(\mu\;\eta)^T}=\det H,$$

where $4N\times 4N$ matrix H is defined as

$$H=\begin{pmatrix} -J+P\left(\hat D^{(\alpha)}\right)^{-1}P^T+P\left(\hat D^{(\alpha+\nu)}\right)^{-1}P^T & P\left(\hat D^{(\alpha)}\right)^{-1}Q^T+P\left(\hat D^{(\alpha+\nu)}\right)^{-1}Q^T\\ Q\left(\hat D^{(\alpha)}\right)^{-1}P^T+Q\left(\hat D^{(\alpha+\nu)}\right)^{-1}P^T & -J+Q\left(\hat D^{(\alpha)}\right)^{-1}Q^T+Q\left(\hat D^{(\alpha+\nu)}\right)^{-1}Q^T\end{pmatrix}.$$

Its $2N\times 2N$ blocks H_{11} and H_{22} coincide with the matrix G given by (6)–(10). This prompts to consider the matrix

$$\tilde H=\begin{pmatrix}R & 0\\ 0 & R\end{pmatrix}H\begin{pmatrix}R & 0\\ 0 & R\end{pmatrix},$$

since after this transformation the blocks $\tilde H_{11}$ and $\tilde H_{22}$ are itself block-diagonal:

$$\tilde H_{11}=\tilde H_{22}=\tilde G=\begin{pmatrix}\frac{A^{(\alpha)}+A^{(\alpha+\nu)}}{2} & 0\\ 0 & \frac{\left(A^{(\alpha)}\right)^{-1}+\left(A^{(\alpha+\nu)}\right)^{-1}}{2}\end{pmatrix}.$$

The other two blocks, $\tilde{H}_{12}$ and $\tilde{H}_{21}$, are exponentially small for large r:

$$\tilde{H}_{12} = \begin{pmatrix} \frac{K^{(\alpha)}+K^{(\alpha+\nu)}}{2} & \frac{M^{(\alpha)}+M^{(\alpha+\nu)}}{2} \\ \frac{M^{(\alpha)}+M^{(\alpha+\nu)}}{2} & \frac{L^{(\alpha)}+L^{(\alpha+\nu)}}{2} \end{pmatrix},$$

$$\tilde{H}_{21} = \begin{pmatrix} \frac{K^{(\alpha)}+K^{(\alpha+\nu)}}{2} & -\frac{M^{(\alpha)}+M^{(\alpha+\nu)}}{2} \\ -\frac{M^{(\alpha)}+M^{(\alpha+\nu)}}{2} & \frac{L^{(\alpha)}+L^{(\alpha+\nu)}}{2} \end{pmatrix},$$

where

$$\begin{aligned} K^{(\alpha)}_{yy'} &= \frac{1}{N} \sum_{p_y}^{(\alpha)} C\left(e^{ip_y}\right) e^{-r\gamma(p_y)+ip_y(y-y')}, \\ L^{(\alpha)}_{yy'} &= \frac{1}{N} \sum_{p_y}^{(\alpha)} C^{-1}\left(e^{ip_y}\right) e^{-r\gamma(p_y)+ip_y(y-y')}, \\ M^{(\alpha)}_{yy'} &= \frac{1}{N} \sum_{p_y}^{(\alpha)} e^{-r\gamma(p_y)+ip_y(y-y')}. \end{aligned}$$

Finally, for the product of correlation functions we obtain

$$\mathfrak{F}_{\alpha,\alpha+\nu}(r) = \det \tilde{H} = \det \begin{pmatrix} \tilde{G} & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{G} \end{pmatrix} = \left(\det \tilde{G}\right)^2 \det\left(\mathbf{1} - \tilde{G}^{-1}\tilde{H}_{12}\,\tilde{G}^{-1}\tilde{H}_{21}\right)$$

$$= \mathfrak{M}^4_{\alpha,\alpha+\nu} \det\left(\mathbf{1} - \tilde{G}^{-1}\tilde{H}_{12}\,\tilde{G}^{-1}\tilde{H}_{21}\right). \tag{16}$$

This shows that the integrals (3) indeed give us the vacuum expectation values. It is also clear that the expansion of (16)

$$\det\left(\mathbf{1} - B\right) = 1 - \operatorname{Tr} B + \frac{1}{2}\left[(\operatorname{Tr} B)^2 - \operatorname{Tr} B^2\right] + \dots,$$

$$B = \tilde{G}^{-1}\tilde{H}_{12}\,\tilde{G}^{-1}\tilde{H}_{21},$$

correspond to form factor representation of the product (15) (nth term of this series contains $2n$ factors of type $e^{-r\gamma(p)}$).

Therefore, to solve the problem of calculation of vacuum expectation values and correlation functions of monodromy fields completely, it remains to find the determinants and inverses of matrices $\frac{A^{(\nu_1)}+A^{(\nu_2)}}{2}$ (see (12)–(14)). It is worth to note that Toeplitz matrices of the same type have recently arisen in the study of free-fermion statistical models [14].

5 One nontrivial case

A simple check of the above formulae can be performed for $\nu = 0$, since in this case both $\mathfrak{M}_{\alpha,\alpha}(r)$ and $\mathfrak{F}_{\alpha,\alpha}(r)$ should be equal to 1 by definition. As for the first quantity, this is an immediate consequence of the representation (12). To prove the same thing for $\mathfrak{F}_{\alpha,\alpha}(r)$, we remark that

$$\mathfrak{F}_{\alpha,\alpha}(r) = \det \left\{ \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} - \begin{pmatrix} \left(A^{(\alpha)}\right)^{-1} & 0 \\ 0 & A^{(\alpha)} \end{pmatrix} \begin{pmatrix} K^{(\alpha)} & M^{(\alpha)} \\ M^{(\alpha)} & L^{(\alpha)} \end{pmatrix} \right.$$
$$\left. \times \begin{pmatrix} \left(A^{(\alpha)}\right)^{-1} & 0 \\ 0 & A^{(\alpha)} \end{pmatrix} \begin{pmatrix} K^{(\alpha)} & -M^{(\alpha)} \\ -M^{(\alpha)} & L^{(\alpha)} \end{pmatrix} \right\}.$$

The matrices $A^{(\alpha)}$, $K^{(\alpha)}$, $L^{(\alpha)}$, $M^{(\alpha)}$ can be simultaneously diagonalized by the discrete Fourier transformation, so that

$$\left(\tilde{G}^{-1}\tilde{H}_{12}\,\tilde{G}^{-1}\tilde{H}_{21}\right)_{qq'} = \frac{1}{N}\sum_{y,y'=0}^{N-1} \left(\tilde{G}^{-1}\tilde{H}_{12}\,\tilde{G}^{-1}\tilde{H}_{21}\right)_{yy'} e^{-iqy+iq'y'}$$
$$= \begin{pmatrix} C^{-1}\left(e^{iq}\right) & 0 \\ 0 & C\left(e^{iq}\right) \end{pmatrix} \begin{pmatrix} C\left(e^{iq}\right) & 1 \\ 1 & C^{-1}\left(e^{iq}\right) \end{pmatrix}$$
$$\times \begin{pmatrix} C^{-1}\left(e^{iq}\right) & 0 \\ 0 & C\left(e^{iq}\right) \end{pmatrix} \begin{pmatrix} C\left(e^{iq}\right) & -1 \\ -1 & C^{-1}\left(e^{iq}\right) \end{pmatrix} e^{-2r\gamma(q)}\,\delta_{qq'} = 0.$$

Consequently, one obtains $\mathfrak{F}_{\alpha,\alpha}(r) = 1$.

There is also one nontrivial case where the explicit form of the answer for the determinants (12) is known. Namely, when $\alpha = 0$ and $\nu = \frac{1}{2}$, the above theory is related to the Ising model on the cylindrical lattice. In particular, for $0 < t < 1$ the correlation function of monodromy fields coincides with the squared correlator of the Ising disorder operators:

$$\langle \mathcal{O}_{0,\frac{1}{2}}(0,0)\mathcal{O}_{\frac{1}{2},0}(r,0)\rangle = \langle \mathcal{O}_{\frac{1}{2},0}(0,0)\mathcal{O}_{0,\frac{1}{2}}(r,0)\rangle = \langle \mu(0,0)\mu(r,0)\rangle^{\,2},$$

and, therefore,

$$\mathfrak{M}_{0,\,\frac{1}{2}} = \left({}_0\langle vac|\mu(0,0)|vac\rangle_{\frac{1}{2}}\right)^2.$$

All Ising form factors on the finite periodic lattice are known [3, 4, 10, 11]. In particular, the vacuum expectation value is given by

$$\left({}_0\langle vac|\mu(0,0)|vac\rangle_{\frac{1}{2}}\right)^2 = \xi\cdot\xi_T\,,$$

where

$$\xi^4 = \frac{\sinh\gamma(0)\sinh\gamma(\pi)}{\sinh\frac{\gamma(0)+\gamma(\pi)}{2}} = \left|1-t^4\right| , \tag{17}$$

$$\xi_T^4 = \frac{\prod_q^{(0)}\prod_p^{(1/2)}\sinh^2\frac{\gamma(q)+\gamma(p)}{2}}{\prod_q^{(0)}\prod_p^{(0)}\sinh\frac{\gamma(q)+\gamma(p)}{2}\prod_q^{(1/2)}\prod_p^{(1/2)}\sinh\frac{\gamma(q)+\gamma(p)}{2}} .$$

Thus we have

$$\det\left(\frac{A^{(0)}+A^{(1/2)}}{2}\right) = \det\left(\frac{\left(A^{(0)}\right)^{-1}+\left(A^{(1/2)}\right)^{-1}}{2}\right) = \xi\cdot\xi_T . \tag{18}$$

Note that this last formula can be checked both analytically and numerically for small values of N. However, it is not clear how it can be obtained directly from the representation (12)–(14), without passing through the solution of the Ising model on the cylinder. A proof of (18), not referring to the Ising model, would give a strong insight on how to obtain the answer for general monodromy.

It should be noted that for $N\to\infty$ one has $\xi_T\to 1$, and the relation (17) reproduces Yang's formula [20] for Ising spontaneous magnetization. We conjecture a similar behaviour for general α, ν: if $0<t<1$, then the quantity $\mathfrak{M}_{\alpha,\alpha+\nu}$ should tend to some finite value as $N\to\infty$. Up to the present moment, this assumption has no strict mathematical proof, and is supported only by a numerical evidence.

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