DYSON'S CONSTANT FOR THE HYPERGEOMETRIC KERNEL

OLEG LISOVYY

Abstract. We study a Fredholm determinant of the hypergeometric kernel arising in the representation theory of the infinite-dimensional unitary group. It is shown that this determinant coincides with the Palmer-Beatty-Tracy tau function of a Dirac operator on the hyperbolic disk. Solution of the connection problem for Painlevé VI equation allows to determine its asymptotic behavior up to a constant factor, for which a conjectural expression is given in terms of Barnes functions. We also present analogous asymptotic results for the Whittaker and Macdonald kernel.

1. Introduction

Connections between Painlevé equations and Fredholm determinants have long been a subject of great interest, mainly because of their applications in random matrix theory and integrable systems, see e.g. [17, 27, 29]. One of the most famous examples is concerned with the Fredholm determinant $F(t) = \det(1 - K_{\mathrm{sine}})$, where K_{sine} is the integral operator with the sine kernel $\frac{\sin(x-y)}{\pi(x-y)}$ on the interval $[0, t]$. It is well-known that $F(t)$ is equal to the gap probability for the Gaussian Unitary Ensemble (GUE) in the bulk scaling limit. As shown in [17], the function $\sigma(t) = t\frac{d}{dt}\ln F(t)$ satisfies the σ-form of a Painlevé V equation,

$$(t\sigma'')^2 + 4\left(t\sigma' - \sigma\right)\left(t\sigma' - \sigma + (\sigma')^2\right) = 0. \tag{1.1}$$

Equation (1.1) and the obvious leading behavior $F(t \to 0) = 1 - t + O\left(t^2\right)$ provide an efficient method of numerical computation of $F(t)$ for all t. Further, as $t \to \infty$, one has

$$F(2t) = f_0\, t^{-\frac14} e^{-t^2/2}\Big(1 + \sum_{k=1}^{N} f_k t^{-k} + O\big(t^{-N-1}\big)\Big).$$

The coefficients $f_1, f_2, \ldots$ in this expansion can in principle be determined from (1.1). It was conjectured by Dyson [12] that the value of the remaining unknown constant is $f_0 = 2^{\frac{1}{12}} e^{3\zeta'(-1)}$, where $\zeta(z)$ is the Riemann ζ-function.

Dyson's conjecture was rigorously proved only recently in [9, 13, 19]. Similar results were also obtained in [3, 8] for the Airy-kernel determinant describing the largest eigenvalue distribution for GUE in the edge scaling limit [26].

The present paper is devoted to the asymptotic analysis of the Fredholm determinant of the hypergeometric kernel on $L^2(0, t)$ with $t \in (0, 1)$. This determinant, to be denoted by $D(t)$, arises in the representation theory of the infinite-dimensional unitary group [5] and provides a 4-parameter class of solutions to Painlevé VI (PVI) equation [6]. Rather surprisingly, it turns out to coincide with the Palmer-Beatty-Tracy (PBT) τ-function of a Dirac operator on the hyperbolic disk [20, 23] under suitable identification of parameters. Relation to PVI allows to give a complete description of the behavior of $D(t)$ as $t \to 1$ up to a constant factor analogous to Dyson's constant f_0 in the sine-kernel asymptotics. Relation to the PBT τ-function, on the other hand, suggests a conjectural expression for this constant in terms of Barnes functions.

The paper is planned as follows. In Section 2, we recall basic facts on Painlevé VI and the associated linear system. The ${}_2F_1$ kernel determinant $D(t)$ and the PBT τ-function are introduced in Sections 3 and 4. Section 5 gives a simple proof of a result of [6], relating $D(t)$ to Painlevé VI. In Section 6, we discuss Jimbo's asymptotic formula for PVI and determine the monodromy corresponding to the ${}_2F_1$ kernel solution. Section 7 contains the main results of the paper: the asymptotics of $D(t)$ as $t \to 1$, obtained from the solution of PVI connection problem (Proposition 7) and a conjecture for the unknown constant (Conjecture 8). Numerical and analytic tests of the conjecture are discussed in Sections 8 and 9. Similar asymptotic results for the Whittaker and Macdonald kernel are presented in Section 10. Appendix A contains a brief summary of formulas for the Barnes function.

2. Painlevé VI and JMU τ-function

Consider the linear system

$$\frac{d\Phi}{d\lambda} = \left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_t}{\lambda - t}\right)\Phi, \tag{2.1}$$

where $A_\nu \in \mathfrak{sl}_2(\mathbb{C})$ $(\nu = 0, 1, t)$ are independent of λ with eigenvalues $\pm\theta_\nu/2$ and

$$A_0 + A_1 + A_t = \begin{pmatrix} -\theta_\infty/2 & 0 \\ 0 & \theta_\infty/2 \end{pmatrix}, \qquad \theta_\infty \neq 0.$$

The fundamental matrix solution $\Phi(\lambda)$ is a multivalued function on $\mathbb{P}^1\backslash\{0, 1, t, \infty\}$. Fix the basis of loops as shown in Fig. 1 and denote by $M_0, M_t, M_1, M_\infty \in SL(2, \mathbb{C})$ the corresponding monodromy matrices. Clearly, one has $M_\infty M_1 M_t M_0 = \mathbf{1}$.

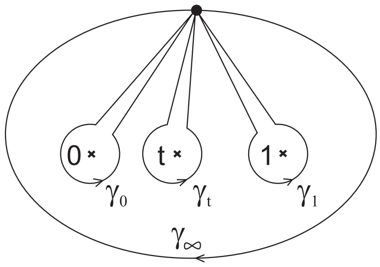

Fig. 1: Generators of $\pi_1\left(\mathbb{P}^1\backslash\{0, 1, t, \infty\}\right)$.

Since the monodromy is defined up to overall conjugation, it is convenient to introduce, following [16], a 7-tuple of invariant quantities

$$p_\nu = \operatorname{Tr} M_\nu = 2\cos\pi\theta_\nu, \qquad \nu = 0, 1, t, \infty, \tag{2.2}$$

$$p_{\mu\nu} = \operatorname{Tr}(M_\mu M_\nu) = 2\cos\pi\sigma_{\mu\nu}, \qquad \mu, \nu = 0, 1, t. \tag{2.3}$$

These data uniquely fix the conjugacy class of the triple (M_0, M_1, M_t) unless the monodromy is reducible. The traces (2.2)–(2.3) satisfy Jimbo-Fricke relation

$$p_{0t}p_{1t}p_{01} + p_{0t}^2 + p_{1t}^2 + p_{01}^2 - (p_0 p_t + p_1 p_\infty)p_{0t} - (p_1 p_t + p_0 p_\infty)p_{1t} - (p_0 p_1 + p_t p_\infty)p_{01} = 4.$$

As a consequence, for fixed $\{p_\nu\}$, p_{0t}, p_{1t} there are at most two possible values for p_{01}.

It is well-known that the monodromy preserving deformations of the system (2.1) are described by the so-called Schlesinger equations

$$\frac{dA_0}{dt} = \frac{[A_t, A_0]}{t}, \qquad \frac{dA_1}{dt} = \frac{[A_t, A_1]}{t - 1}, \tag{2.4}$$

which are equivalent to the sixth Painlevé equation:

$$\begin{aligned} \frac{d^2q}{dt^2} = {} & \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\frac{dq}{dt} + \\ & + \frac{q(q-1)(q-t)}{2t^2(t-1)^2}\left((\theta_\infty - 1)^2 - \frac{\theta_0^2 t}{q^2} + \frac{\theta_1^2(t-1)}{(q-1)^2} + \frac{(1-\theta_t^2)t(t-1)}{(q-t)^2}\right). \end{aligned} \tag{2.5}$$

Relation between $A_{0,1,t}(t)$ and $q(t)$ is given by $\left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_t}{\lambda - t}\right)_{12} = \frac{k(t)(\lambda - q(t))}{\lambda(\lambda-1)(\lambda-t)}$.

Jimbo-Miwa-Ueno (JMU) τ-function [18] of Painlevé VI is defined as follows:

$$\frac{d}{dt}\ln\tau_{JMU}(t;\boldsymbol{\theta})=\frac{\operatorname{tr}(A_0A_t)}{t}+\frac{\operatorname{tr}(A_1A_t)}{t-1}\,, \tag{2.6}$$

where $\boldsymbol{\theta}=(\theta_0,\theta_1,\theta_t,\theta_\infty)$. Introducing a logarithmic derivative

$$\sigma(t)=t(t-1)\frac{d}{dt}\ln\tau_{JMU}(t;\boldsymbol{\theta})+\frac{t\left(\theta_t^2-\theta_\infty^2\right)}{4}-\frac{\theta_t^2+\theta_0^2-\theta_1^2-\theta_\infty^2}{8}, \tag{2.7}$$

it can be deduced from the Schlesinger system (2.4) that $\sigma(t)$ satisfies the following 2nd order ODE (σ-form of Painlevé VI):

$$\sigma'\Big(t(t-1)\sigma''\Big)^2+\left[2\sigma'(t\sigma'-\sigma)-(\sigma')^2-\frac{(\theta_t^2-\theta_\infty^2)(\theta_0^2-\theta_1^2)}{16}\right]^2= \tag{2.8}$$
$$=\left(\sigma'+\frac{(\theta_t+\theta_\infty)^2}{4}\right)\left(\sigma'+\frac{(\theta_t-\theta_\infty)^2}{4}\right)\left(\sigma'+\frac{(\theta_0+\theta_1)^2}{4}\right)\left(\sigma'+\frac{(\theta_0-\theta_1)^2}{4}\right).$$

In terms of $q(t)$, the definition of $\sigma(t)$ reads

$$\sigma(t)=\frac{t^2(t-1)^2}{4q(q-1)(q-t)}\left(q'-\frac{q(q-1)}{t(t-1)}\right)^2-\frac{\theta_0^2t}{4q}+\frac{\theta_1^2(t-1)}{4(q-1)}-\frac{\theta_t^2t(t-1)}{4(q-t)} \tag{2.9}$$
$$-\frac{\theta_\infty^2(q-1)}{4}-\frac{\theta_t^2t}{4}+\frac{\theta_t^2+\theta_0^2-\theta_1^2-\theta_\infty^2}{8}.$$

3. Hypergeometric kernel determinant

It was shown in [5] that the spectral measure associated to the decomposition of a remarkable 4-parameter family of characters of the infinite-dimensional unitary group $U(\infty)$ gives rise to a determinantal point process with correlation kernel

$$K(x,y)=\lambda\,\frac{A(x)B(y)-B(x)A(y)}{y-x},\qquad x,y\in(0,1),$$

where

$$\lambda=\frac{\sin\pi z\sin\pi z'}{\pi^2}\Gamma\left[\begin{array}{c}1+z+w,1+z+w',1+z'+w,1+z'+w'\\1+z+z'+w+w',2+z+z'+w+w'\end{array}\right], \tag{3.1}$$

$$A(x)=x^{\frac{z+z'+w+w'}{2}}(1-x)^{-\frac{z+z'+2w'}{2}}{}_2F_1\left[\begin{array}{c}z+w',z'+w'\\z+z'+w+w'\end{array}\Big|\,\frac{x}{x-1}\right], \tag{3.2}$$

$$B(x)=x^{\frac{z+z'+w+w'+2}{2}}(1-x)^{-\frac{z+z'+2w'+2}{2}}{}_2F_1\left[\begin{array}{c}z+w'+1,z'+w'+1\\z+z'+w+w'+2\end{array}\Big|\,\frac{x}{x-1}\right]. \tag{3.3}$$

Note that our notation slightly differs from the standard one [5, 6]; to shorten some formulas from Painlevé theory, the interval $\left(\frac12,\infty\right)$ of [5, 6] is mapped to $(0,1)$ by $x\mapsto 1/\left(\frac12+x\right)$.

The kernel $K(x,y)$ has a number of symmetries:

(S1) It is invariant under transformations $z\leftrightarrow z'$ and $w\leftrightarrow w'$; the latter symmetry follows from
$${}_2F_1\left[\begin{array}{c}a,b\\c\end{array}\Big|\,z\right]=(1-z)^{c-a-b}{}_2F_1\left[\begin{array}{c}c-a,c-b\\c\end{array}\Big|\,z\right].$$

(S2) It is also straightforward to check that $K(x,y)$ is invariant under transformation
$$z\mapsto -z,\ z'\mapsto -z',\ w\mapsto w'+z+z',\ w'\mapsto w+z+z'.$$

(S3) We can simultaneously shift $z\mapsto z\pm1$, $z'\mapsto z'\pm1$, $w\mapsto w\mp1$, $w'\mapsto w'\mp1$; together with (S2), this allows to assume without loss of generality that $0\le\operatorname{Re}(z+z')\le1$.

We are interested in the Fredholm determinant

$$D(t)=\det\left(1-K\big|_{(0,t)}\right),\qquad t\in(0,1)\,. \tag{3.4}$$

Assume that the parameters $z,z',w,w'\in\mathbb{C}$ satisfy the conditions:

(C1) $z'=\bar z\in\mathbb{C}\backslash\mathbb{Z}$ or $k<z,z'<k+1$ for some $k\in\mathbb{Z}$,

(C2) $w' = \bar{w} \in \mathbb{C}\backslash\mathbb{Z}$ or $l < w, w' < l+1$ for some $l \in \mathbb{Z}$,
(C3) $z + z' + w + w' > 0$, $|z + z'| < 1$, $|w + w'| < 1$.

Then, as was shown by Borodin and Deift in [6], the determinant (3.4) is well-defined and $D(t) = \tau_{JMU}(t; \boldsymbol{\theta})$ for the following choice of PVI parameters:

$$\boldsymbol{\theta} = (z + z' + w + w', z - z', 0, w - w'). \tag{3.5}$$

The original proof in [6] that $D(t)$ satisfies Painlevé VI is rather involved. In Section 5, we give an alternative simple derivation of this result in the spirit of [27].

Lemma 1. *Assume (C1)–(C3). Then the asymptotic expansion of $D(t)$ as $t \to 0$ has the form*

$$D(t) = 1 - \kappa \cdot t^{1+z+z'+w+w'} + O\left(t^{2+z+z'+w+w}\right), \tag{3.6}$$

where

$$\kappa = \frac{\sin \pi z \sin \pi z'}{\pi^2} \Gamma \left[\begin{array}{c} 1+z+w, 1+z+w', 1+z'+w, 1+z'+w' \\ 2+z+z'+w+w', 2+z+z'+w+w' \end{array} \right]. \tag{3.7}$$

Proof. As $t \to 0$, one has $D(t) \sim 1 - \int_0^t K(x,x)\,dx$. The result then follows from

$$A(x) \sim x^{\frac{z+z'+w+w'}{2}}, \qquad B(x) \sim x^{\frac{z+z'+w+w'}{2}+1} \qquad \text{as } x \to 0.$$

Note that in the expression for κ given in Remark 7.2 in [6] the gamma product is missing, which seems to be a typesetting error. □

The asymptotics (3.6) and σPVI equation (2.8) uniquely fix $D(t)$ by a result of [7]. Gamma product in (3.7) is a function of θ_0, θ_1, θ_∞ only, but $\frac{\sin \pi z \sin \pi z'}{\pi^2}$ depends on an additional parameter (e.g. $z + z'$); hence we are dealing with a 1-parameter family of initial conditions.

The results of [6] can be extended to a larger set of parameters. This follows already from the observation that the subset of $\mathbb{C}^4$ defined by (C1)–(C3) is not stable under the transformations (S1)–(S3). However, instead of trying to identify all admissible values of z, z', w, w', in the remainder of this paper we simply replace (C1)–(C3) by a much weaker (invariant) condition

(C4) $z + w, z + w', z' + w, z' + w' \notin \mathbb{Z}_{<0}$ and $\mathrm{Re}\,(z + z' + w + w') > 0$,

and *define* $D(t)$ as the JMU τ-function of Painlevé VI with parameters (3.5), whose leading behavior as $t \to 0$ is specified by (3.6)–(3.7). Our aim in the next sections is to determine the asymptotics of $D(t)$ as $t \to 1$.

4. PBT τ-function

Palmer-Beatty-Tracy τ-function [20, 23] is a regularized determinant of the quantum hamiltonian of a massive Dirac particle moving on the hyperbolic disk in the superposition of a uniform magnetic field B and the field of two non-integer Aharonov-Bohm fluxes $2\pi\nu_{1,2}$ $(-1 < \nu_{1,2} < 0)$ located at the points $a_{1,2}$.

Denote by m and E the particle mass and energy, by $-4/R^2$ the disk curvature and write $b = \frac{BR^2}{4}$, $\mu = \frac{\sqrt{(m^2-E^2)R^2+4b^2}}{2}$, $s = \tanh^2 \frac{d(a_1,a_2)}{R}$, where $d(a_1, a_2)$ denotes the geodesic distance between a_1 and a_2. Then $\tau_{PBT}(s)$ can be expressed [23] in terms of a solution $u(s)$ of the sixth Painlevé equation (2.5):

$$\begin{aligned} \frac{d}{ds} \ln \tau_{PBT}(s) = {} & \frac{s(1-s)}{4u(1-u)(u-s)} \left(\frac{du}{ds} - \frac{1-u}{1-s} \right)^2 \\ & - \frac{1-u}{1-s} \left(\frac{(\theta_\infty - 1)^2}{4s} - \frac{(\theta_0+1)^2}{4u} + \frac{\theta_t^2}{4(u-s)} \right), \end{aligned} \tag{4.1}$$

where the corresponding PVI parameters are given by

$$\boldsymbol{\theta} = (1 + \nu_1 + \nu_2 - 2b, 0, 2\mu, 1 + \nu_1 - \nu_2).$$

The initial conditions are specified by the asymptotics of $\tau_{PBT}(s)$ as $s \to 1$, computed in [20]:

$$\tau_{PBT}(s) = 1 - \kappa_{PBT}(1-s)^{1+2\mu} + O\left((1-s)^{2+2\mu}\right), \tag{4.2}$$

$$\kappa_{PBT} = \frac{\sin \pi\nu_1 \sin \pi\nu_2}{\pi^2} \Gamma\left[\begin{array}{c} 2+\mu+\nu_1-b, \mu-\nu_1+b, 2+\mu+\nu_2-b, \mu-\nu_2+b \\ 2+2\mu, 2+2\mu \end{array} \right].$$

Some resemblance between (2.9) and (4.1) suggests that $\tau_{PBT}(s)$ is a special case of the JMU τ-function. Indeed, consider the following transformation:

$$s \mapsto 1-t, \qquad u \mapsto \frac{1-t}{1-q}.$$

In the notation of Table 1 of [21], this corresponds to Bäcklund transformation $r_x P_{xy}$ for Painlevé VI. If $u(s)$ is a solution with parameters $\boldsymbol{\theta} = (\theta_0, \theta_1, \theta_t, \theta_\infty)$, then $q(t)$ solves PVI with parameters $\boldsymbol{\theta'} = (\theta_t, \theta_\infty - 1, \theta_1, \theta_0 + 1)$. Straightforward calculation then shows that $\tau_{PBT}(1-t) = \tau_{JMU}(t; \boldsymbol{\theta'})$ provided $\theta_1 = 0$.

Lemma 2. *Under the following identification of parameters*

$$z + z' + w + w' = 2\mu, \qquad z - z' = \nu_1 - \nu_2, \qquad w - w' = 2 + \nu_1 + \nu_2 - 2b, \tag{4.3}$$

$$\cos \pi(z + z') = \cos \pi(\nu_1 + \nu_2), \tag{4.4}$$

we have $D(t) = \tau_{PBT}(1-t)$.

Proof. It was shown above that if (4.3) holds, then both $D(t)$ and $\tau_{PBT}(1-t)$ are JMU τ-functions with the same $\boldsymbol{\theta}$. To show the equality, it suffices to verify that (4.4) implies $\kappa = \kappa_{PBT}$. □

Symmetries of $D(t)$ imply that $\tau_{PBT}(s)$ is invariant under transformations

(S1) $\mu \mapsto \mu$, $\nu_{1,2} \mapsto \nu_{1,2}$, $b \mapsto 2 + \nu_1 + \nu_2 - b$;

(S2) $\mu \mapsto \mu$, $\nu_{1,2} \mapsto -2 - \nu_{1,2}$, $b \mapsto -b$.

These symmetries of $\tau_{PBT}(s)$ are by no means manifest, although they can also be deduced from the Fredholm determinant representation in [20], Theorem 1.1.

5. Painlevé VI from Tracy-Widom equations

5.1. **Basic notation.** Tracy and Widom [27] have developed a systematic approach for deriving differential equations satisfied by Fredholm determinants of the form

$$D_I = \det\left(1 - K_I\right), \tag{5.1}$$

where K_I is an integral operator with the kernel

$$K_I(x,y) = \frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x-y}, \tag{5.2}$$

on $L^2(J)$, with $J = \bigcup_{j=1}^{M} (a_{2j-1}, a_{2j})$. The kernels of the form (5.2) are called integrable and possess rather special properties: e.g. it was observed in [15] that the kernel of the resolvent $\left(1-K_I\right)^{-1} K_I$ is also integrable.

The method of [27] requires that φ, ψ in (5.2) obey a system of linear ODEs of the form

$$m(x)\frac{d}{dx}\left(\begin{array}{c} \varphi \\ \psi \end{array}\right) = \left(\sum_{k=0}^{N} \mathcal{A}_k x^k\right)\left(\begin{array}{c} \varphi \\ \psi \end{array}\right), \tag{5.3}$$

where $m(x)$ is a polynomial and $\mathcal{A}_k \in \mathfrak{sl}_2(\mathbb{C})$ $(k = 0, \ldots, N)$. Note that a linear transformation $\left(\begin{array}{c} \varphi \\ \psi \end{array}\right) \mapsto G\left(\begin{array}{c} \varphi \\ \psi \end{array}\right)$ leaves $K_I(x,y)$ invariant provided $\det G = 1$, and therefore $\{\mathcal{A}_k\}$ can be conjugated by an arbitrary $SL(2,\mathbb{C})$-matrix.

Our aim is to show that in the special case

$$m(x) = x(1-x), \qquad N = 1, \qquad J = (0,t)$$

the determinant (5.1) (i) coincides with the ${}_2F_1$ kernel determinant $D(t)$ and (ii) considered as a function of t, is a Painlevé VI τ-function.

Let us temporarily switch to the notation of [27] and introduce the quantities

$$q(t) = \left[(1-K_I)^{-1}\varphi\right](t), \qquad p(t) = \left[(1-K_I)^{-1}\psi\right](t),$$

$$u(t) = \langle\varphi|(1-K_I)^{-1}|\varphi\rangle, \qquad v(t) = \langle\varphi|(1-K_I)^{-1}|\psi\rangle, \qquad w(t) = \langle\psi|(1-K_I)^{-1}|\psi\rangle,$$

where the inner products $\langle\,|\,\rangle$ are taken over J. Then

$$D_I^{-1}D_I{}' = qp' - pq',$$

with primes denoting derivatives with respect to t. Tracy-Widom approach gives a system of nonlinear first order ODEs for q, p, u, v, w, which we are about to examine.

5.2. **Derivation.** Let $\mathcal{A}_1$ be diagonalizable, so that one can set

$$\mathcal{A}_0 = \begin{pmatrix} \alpha_0 & \beta_0 \\ -\gamma_0 & -\alpha_0 \end{pmatrix}, \qquad \mathcal{A}_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & -\alpha_1 \end{pmatrix}.$$

The Tracy-Widom equations then read

$$t(1-t)\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}, \tag{5.4}$$

$$u' = q^2, \qquad v' = pq, \qquad w' = p^2, \tag{5.5}$$

where

$$\alpha = \alpha_0 + \alpha_1 t + v, \qquad \beta = \beta_0 + (2\alpha_1 - 1)u, \qquad \gamma = \gamma_0 - (2\alpha_1+1)w.$$

The system (5.4)–(5.5) has two first integrals

$$I_1 = 2\alpha pq + \beta p^2 + \gamma q^2 - 2\alpha_1 v, \tag{5.6}$$

$$I_2 = (v+\alpha_0)^2 - \beta\gamma - 2\alpha_1 t(1-t)pq + 2\alpha_1(1-t)v - I_1 t. \tag{5.7}$$

Consider the logarithmic derivative $\zeta(t) = t(t-1)D_I^{-1}D_I{}'$. It can be easily checked that

$$\zeta = 2\alpha pq + \beta p^2 + \gamma q^2 = 2\alpha_1 v + I_1, \tag{5.8}$$

$$\zeta' = 2\alpha_1 pq, \tag{5.9}$$

$$t(1-t)\zeta'' = 2\alpha_1(\beta p^2 - \gamma q^2). \tag{5.10}$$

Note that v, α are expressible in terms of ζ and pq in terms of ζ'. Using (5.7) and (5.8) one may also write $\beta\gamma$ and $\beta p^2 + \gamma q^2$ in terms of ζ and ζ'. Now squaring (5.10) we find a second order equation for ζ:

$$\left(t(1-t)\zeta''\right)^2 + 4\left(\zeta' - \alpha_1^2\right)\left(t\zeta' - \zeta\right)^2 - 4\zeta'\left(t\zeta' - \zeta\right)\left(\zeta' + 2\alpha_0\alpha_1 - I_1\right) = 4(I_1 + I_2)\left(\zeta'\right)^2. \tag{5.11}$$

If we parameterize the integrals I_1, I_2 as

$$I_1 = -k_1k_2 + \alpha_1(2\alpha_0 + \alpha_1),$$

$$I_2 = \frac{(k_1+k_2)^2}{4} - \alpha_1(2\alpha_0 + \alpha_1),$$

and define

$$\sigma(t) = \zeta(t) - \alpha_1^2 t + \frac{\alpha_1^2 + k_1k_2}{2}, \tag{5.12}$$

then (5.11) transforms into σPVI equation (2.8) with parameters $\boldsymbol{\theta} = (k_1 - k_2, k_1 + k_2, 0, 2\alpha_1)$. Moreover, (5.12) and the definition of $\zeta(t)$ imply that $D_I(t)$ coincides with the corresponding JMU τ-function.

The system (5.3) has two linearly independent solutions, only one of which can be chosen to be regular as $x \to 0$. This is the only solution of interest here, as if φ, ψ have an irregular part, the operator K_I fails to be trace-class. The regularity further implies that q, p, u, v, w vanish as $t \to 0$, and therefore the integrals I_1, I_2 are given by

$$I_1 = 0, \qquad I_2 = \alpha_0^2 - \beta_0\gamma_0.$$

Choosing $\mathcal{A}_0$, $\mathcal{A}_1$ as above, one can still conjugate them by a diagonal matrix. Use this freedom to parameterize α_0, β_0, γ_0, α_1 as follows:

$$\begin{aligned}
\alpha_0 &= -\frac{c}{2} - \frac{ab}{c-a-b}, \\
\beta_0 &= -\frac{(c-a)(c-b)}{c-a-b}, \\
\gamma_0 &= -\frac{ab}{c-a-b}, \\
\alpha_1 &= \frac{c-a-b}{2},
\end{aligned}$$

so that $I_2 = \dfrac{c^2}{4}$ and therefore $(k_1+k_2)^2 = (a-b)^2$, $(k_1-k_2)^2 = c^2$. Now if $\operatorname{Re} c > 0$, the regular solution of (5.3) is given by
(5.13)

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}(x) = \text{const}\cdot \begin{pmatrix} 1 & \frac{(c-a)(c-b)}{c(1+c)} \\ -1 & -\frac{ab}{c(1+c)} \end{pmatrix} \begin{pmatrix} x^{\frac{c}{2}}(1-x)^{-\frac{a+b}{2}} \; {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}\,\middle|\, \frac{x}{x-1}\right] \\ x^{1+\frac{c}{2}}(1-x)^{-1-\frac{a+b}{2}} \; {}_2F_1\left[\begin{matrix} 1+a, 1+b \\ 2+c \end{matrix}\,\middle|\, \frac{x}{x-1}\right] \end{pmatrix}.$$

Setting $a = z+w'$, $b = z'+w'$, $c = z+z'+w+w'$ and comparing (5.13) with (3.2)–(3.3) we see that $K_I(t)$ coincides, up to an adjustable constant factor, with the ${}_2F_1$ kernel of Section 3.

Remark 3. A system similar to (5.4)–(5.5) has already appeared in the Tracy-Widom analysis of the Jacobi kernel, see Section V.C of [27]. As the integral I_2 was not noticed there, the final result of [27] was a *third* order ODE (as one may well guess, it represents the first derivative of (5.11) in a disguised form). Later Haine and Semengue [14] have derived another third order equation for the Jacobi kernel determinant using the Virasoro approach of [2], and obtained Painlevé VI as the compatibility condition of the two equations. Our calculation gives, among other things, an elementary proof of this result.

Remark 4. For non-diagonalizable $\mathcal{A}_1$ it can be assumed that $\mathcal{A}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The equations (5.5) remain unchanged, whereas instead of (5.4) we get

$$t(1-t)\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} \tilde\alpha & \tilde\beta \\ -\tilde\gamma & -\tilde\alpha \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix},$$

where

$$\tilde\alpha = \alpha_0 + v - w, \qquad \tilde\beta = \beta_0 + s - u + 2v, \qquad \tilde\gamma = \gamma_0 - w.$$

As before, we have two first integrals,

$$\begin{aligned}
I_1 &= 2\tilde\alpha pq + \tilde\beta p^2 + \tilde\gamma(q^2+1), \\
I_2 &= \tilde\alpha^2 - \tilde\beta\tilde\gamma - t(1-t)p^2 + (2t-1)\tilde\gamma - I_1 t.
\end{aligned}$$

The rest of the computation is completely analogous to the diagonalizable case. As a final result, one finds that the determinant $D(t)$ with $w = w'$ is a τ-function of Painlevé VI with parameters $\boldsymbol{\theta} = (z+z'+2w, z-z', 0, 0)$.

6. Jimbo's asymptotic formula

A remarkable result of Jimbo [16] relates the asymptotic behavior of the JMU τ-function (2.6) near the singular points $t=0,1,\infty$ to the monodromy of the associated linear system (2.1).

Theorem 5 (Theorem 1.1 in [16])**.** *Assume that*

(J1) $$\theta_0,\theta_1,\theta_t,\theta_\infty\notin\mathbb{Z},$$

(J2) $$0\leq\operatorname{Re}\sigma_{0t}<1,$$

(J3) $$\theta_0\pm\theta_t\pm\sigma_{0t},\theta_\infty\pm\theta_1\pm\sigma_{0t}\notin 2\mathbb{Z}.$$

Then $\tau_{JMU}(t)$ has the following asymptotic expansion as $t\to 0$:

$$\begin{aligned}\tau_{JMU}(t)=\mathrm{const}\cdot t^{\frac{\sigma_{0t}^2-\theta_0^2-\theta_t^2}{4}}\Big[1&-\frac{\left(\theta_0^2-(\theta_t-\sigma_{0t})^2\right)\left(\theta_\infty^2-(\theta_1-\sigma_{0t})^2\right)}{16\sigma_{0t}^2(1+\sigma_{0t})^2}\,\hat{s}\,t^{1+\sigma_{0t}}\\ &-\frac{\left(\theta_0^2-(\theta_t+\sigma_{0t})^2\right)\left(\theta_\infty^2-(\theta_1+\sigma_{0t})^2\right)}{16\sigma_{0t}^2(1-\sigma_{0t})^2}\,\hat{s}^{-1}\,t^{1-\sigma_{0t}}\\ &+\frac{(\theta_0^2-\theta_t^2-\sigma_{0t}^2)(\theta_\infty^2-\theta_1^2-\sigma_{0t}^2)}{8\sigma_{0t}^2}\,t+O\left(|t|^{2(1-\operatorname{Re}\sigma_{0t})}\right)\Big],\end{aligned}\tag{6.1}$$

where $\sigma_{0t}\neq 0$ and

$$\hat{s}=\Gamma\left[\begin{array}{c}1-\sigma_{0t},1-\sigma_{0t},1+\frac{\theta_0+\theta_t+\sigma_{0t}}{2},1-\frac{\theta_0-\theta_t-\sigma_{0t}}{2},1+\frac{\theta_\infty+\theta_1+\sigma_{0t}}{2},1-\frac{\theta_\infty-\theta_1-\sigma_{0t}}{2}\\ 1+\sigma_{0t},1+\sigma_{0t},1+\frac{\theta_0+\theta_t-\sigma_{0t}}{2},1-\frac{\theta_0-\theta_t+\sigma_{0t}}{2},1+\frac{\theta_\infty+\theta_1-\sigma_{0t}}{2},1-\frac{\theta_\infty-\theta_1+\sigma_{0t}}{2}\end{array}\right]s,$$

$$\begin{aligned}&s^{\pm1}\big(\cos\pi(\theta_t\mp\sigma_{0t})-\cos\pi\theta_0\big)\big(\cos\pi(\theta_1\mp\sigma_{0t})-\cos\pi\theta_\infty\big)=\\ &\quad=\left(\pm i\sin\pi\sigma_{0t}\cos\pi\sigma_{1t}-\cos\pi\theta_t\cos\pi\theta_\infty-\cos\pi\theta_0\cos\pi\theta_1\right)e^{\pm i\pi\sigma_{0t}}\\ &\qquad\pm i\sin\pi\sigma_{0t}\cos\pi\sigma_{01}+\cos\pi\theta_t\cos\pi\theta_1+\cos\pi\theta_\infty\cos\pi\theta_0.\end{aligned}$$

If $\sigma_{0t}=0$, then

$$\begin{aligned}\tau_{JMU}(t)=\mathrm{const}\cdot t^{-\frac{\theta_0^2+\theta_t^2}{4}}\Big[1&-\frac{\theta_1\theta_t}{2}\,t-\frac{(\theta_\infty^2-\theta_1^2)(\theta_0^2-\theta_t^2)}{16}\,t(\Omega^2+2\Omega+3)\\ &+\frac{\theta_t(\theta_\infty^2-\theta_1^2)+\theta_1(\theta_0^2-\theta_t^2)}{4}\,t(\Omega+1)+o(|t|)\Big],\end{aligned}$$

where $\Omega=1-\hat{s}'-\ln t$ and

$$\hat{s}'=s'+\psi\left(1+\frac{\theta_0+\theta_t}{2}\right)+\psi\left(1+\frac{\theta_t-\theta_0}{2}\right)+\psi\left(1+\frac{\theta_\infty+\theta_1}{2}\right)+\psi\left(1+\frac{\theta_1-\theta_\infty}{2}\right)-4\psi(1).$$

Here $\psi(x)$ denotes the digamma function and

$$s'=i\pi\,\frac{\cos\pi\sigma_{1t}+\cos\pi\sigma_{01}-\cos\pi\theta_0\,e^{i\pi\theta_1}-\cos\pi\theta_\infty\,e^{i\pi\theta_t}+i\sin\pi(\theta_1+\theta_t)}{\big(\cos\pi\theta_t-\cos\pi\theta_0\big)\big(\cos\pi\theta_1-\cos\pi\theta_\infty\big)}.$$

When one tries to determine from Theorem 5 the monodromy associated to the ${}_2F_1$ kernel solution $D(t)$ of σPVI, it turns out that all three assumptions (J1)–(J3) are not satisfied:

- Firstly, (J1) does not hold since in our case $\theta_t=0$. This requirement can nevertheless be relaxed as the appropriate non-resonancy condition for (2.1) is $\theta_0,\theta_1,\theta_t,\theta_\infty\notin\mathbb{Z}\backslash\{0\}$. The proof of asymptotic formulas when some θ's are equal to zero differs from that in [16] only in technical details; see e.g. [11].
- If we blindly accept (6.1) then from $D(t\to 0)\sim 1$ follows $\sigma_{0t}=\theta_0=z+z'+w+w'$. Thus (J2) is violated unless $\operatorname{Re}\theta_0<1$ and (J3) does not hold in any case. Note, however, that (6.1) admits a well-defined limit as $\theta_t=0$, $\sigma_{0t}\to\theta_0$. In this limit, the coefficients of t and

$t^{1-\sigma_{0t}}$ vanish; we also have

$$\cos\pi\sigma_{01} \to \cos\pi\theta_\infty + (\cos\pi\theta_1 - \cos\pi\sigma_{1t})\, e^{-i\pi\theta_0},$$

$$s(\theta_0 - \sigma_{0t}) \to \frac{1}{\pi}\cdot\frac{\sin\pi\theta_0\,(\cos\pi\theta_1 - \cos\pi\sigma_{1t})}{\sin\frac{\pi}{2}(\theta_\infty - \theta_0 + \theta_1)\sin\frac{\pi}{2}(\theta_\infty + \theta_0 - \theta_1)},$$

and hence the coefficient of $t^{1+\sigma_{0t}}$ becomes

$$\frac{\cos\pi\theta_1 - \cos\pi\sigma_{1t}}{2\pi^2}\,\Gamma\left[\begin{array}{c} 1+\frac{\theta_0+\theta_1+\theta_\infty}{2}, 1+\frac{\theta_0+\theta_1-\theta_\infty}{2}, 1+\frac{\theta_0-\theta_1+\theta_\infty}{2}, 1+\frac{\theta_0-\theta_1-\theta_\infty}{2} \\ 2+\theta_0, 2+\theta_0 \end{array}\right]. \tag{6.2}$$

- Suppose that in our case the error estimate in (6.1) can be improved to $O\left(t^{2+\theta_0}\right)$ (or at least to $o\left(t^{1+\theta_0}\right)$). Then, assuming that $0 \le \operatorname{Re}(z+z') \le 1$ and comparing (6.2) with (3.7), (3.5) one would conclude that $\sigma_{1t} = z + z'$.

The above steps can indeed be justified — after some tedious analysis going into the depths of Jimbo's proof. Alternatively, the monodromy can be extracted from Sections 3, 4 of [6], where σPVI equation for $D(t)$ has itself been derived from a Riemann-Hilbert problem.

7. Asymptotics of $D(t)$ as $t \to 1$

Once the monodromy is known, the asymptotics of $\tau_{JMU}(t)$ as $t \to 1$ can be determined from Jimbo's formula after substitutions $t \leftrightarrow 1-t$, $\theta_0 \leftrightarrow \theta_1$, $\sigma_{0t} \leftrightarrow \sigma_{1t}$, $\sigma_{01} \to \tilde{\sigma}_{01}$, where

$$2\cos\pi\tilde{\sigma}_{01} = \operatorname{Tr}\left(M_0 M_t^{-1} M_1 M_t\right) = p_0 p_1 + p_t p_\infty - p_{0t} p_{1t} - p_{01}. \tag{7.1}$$

Remark 6. The transformation $\sigma_{01} \to \tilde{\sigma}_{01}$ is missing in [16] due to an incorrectly stated symmetry: the relation $\tau_{JMU}(1-t; M_0, M_t, M_1) = \text{const}\cdot\tau_{JMU}(t; M_1, M_t, M_0)$ on p. 1144 of [16] should be replaced by

$$\tau_{JMU}(1-t; M_0, M_t, M_1) = \text{const}\cdot\tau_{JMU}\left(t; (M_t M_0)^{-1} M_1 M_t M_0, (M_0)^{-1} M_t M_0, M_0\right),$$

which can be understood from Fig. 2.

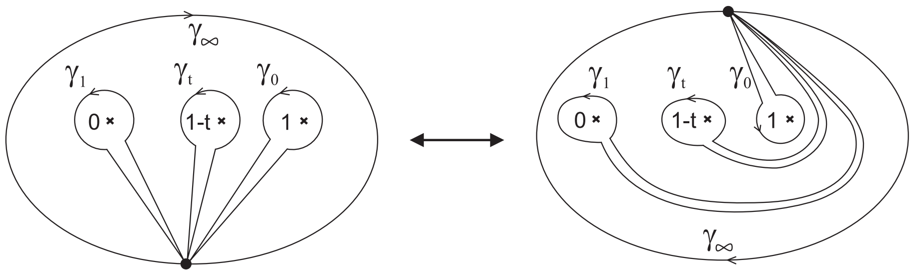

Fig. 2: Homotopy basis after transformation $\lambda \mapsto 1-\lambda$, $t \mapsto 1-t$.

Proposition 7. *Assume that* $0 \le \operatorname{Re}(z+z') < 1$ *and*

$$z, z', w, w', z+z'+w, z+z'+w' \notin \mathbb{Z}.$$

(1) *If* $z+z' \neq 0$, *then the following asymptotics is valid as* $t \to 1$*:*

$$D(t) = C\,(1-t)^{zz'}\Bigg[1 + \frac{zz'\left((z+z'+w)(z+z'+w') + ww'\right)}{(z+z')^2}(1-t) - a^+(1-t)^{1+z+z'} - a^-(1-t)^{1-z-z'} + O\left((1-t)^{2-2\operatorname{Re}(z+z')}\right)\Bigg], \tag{7.2}$$

where C *is a constant and*

$$a^\pm = \Gamma\left[\begin{array}{c} \mp z \mp z', \mp z \mp z', 1 \pm z, 1 \pm z', 1+w+\frac{z+z'}{2} \pm \frac{z+z'}{2}, 1+w'+\frac{z+z'}{2} \pm \frac{z+z'}{2} \\ 2 \pm z \pm z', 2 \pm z \pm z', \mp z, \mp z', w+\frac{z+z'}{2} \mp \frac{z+z'}{2}, w'+\frac{z+z'}{2} \mp \frac{z+z'}{2} \end{array}\right].$$

(2) *Similarly, if $z+z'=0$, then*

$$D(t)=C(1-t)^{-z^2}\Big[1+z^2ww'(1-t)(\tilde\Omega^2+2\tilde\Omega+3)$$
$$+z^2(w+w')(1-t)(\tilde\Omega+1)+o(1-t)\Big], \tag{7.3}$$

where $\tilde\Omega=1-a'-\ln(1-t)$ and

$$a'=\psi(1+z)+\psi(1-z)+\psi(1+w)+\psi(1+w')-4\psi(1).$$

Proof. Take into account that in our case $\theta_t=0$, $\sigma_{0t}=\theta_0$ and replace $\theta_0\leftrightarrow\theta_1$, $\sigma_{0t}\leftrightarrow\sigma_{1t}$, $\sigma_{01}\to\tilde\sigma_{01}$. Different quantities in Theorem 5 then transform into

$$s\to s_{1t}=1\,,\qquad s'\to s'_{1t}=0,$$

$$\hat s\to\hat s_{1t}=\Gamma\left[\begin{array}{c}1-\sigma_{1t},1-\sigma_{1t},1+\frac{\theta_1+\sigma_{1t}}{2},1-\frac{\theta_1-\sigma_{1t}}{2},1+\frac{\theta_0+\theta_\infty+\sigma_{1t}}{2},1+\frac{\theta_0-\theta_\infty+\sigma_{1t}}{2}\\ 1+\sigma_{1t},1+\sigma_{1t},1+\frac{\theta_1-\sigma_{1t}}{2},1-\frac{\theta_1+\sigma_{1t}}{2},1+\frac{\theta_0+\theta_\infty-\sigma_{1t}}{2},1+\frac{\theta_0-\theta_\infty-\sigma_{1t}}{2}\end{array}\right],$$

$$\hat s'\to\hat s'_{1t}=\psi\left(1+\frac{\theta_1}{2}\right)+\psi\left(1-\frac{\theta_1}{2}\right)+\psi\left(1+\frac{\theta_0+\theta_\infty}{2}\right)+\psi\left(1+\frac{\theta_0-\theta_\infty}{2}\right)-4\psi(1).$$

The statement now follows from $\sigma_{1t}=z+z'$ and (3.5). □

The constant C in (7.2)–(7.3) remains as yet undetermined. We will find an expression for it using Lemma 2 and earlier results of Doyon [10], who conjectured that for vanishing magnetic field $\tau_{PBT}(s)$ coincides with a correlation function of twist fields in the theory of free massive Dirac fermions on the hyperbolic disk. The asymptotics of $\tau_{PBT}(s)$ as $s\to0$ and $s\to1$ is then fixed, respectively, by conformal behavior of the correlator and its form factor expansion. The basic statement of [10] (supported by numerics) is that there indeed exists a solution of the appropriate σPVI equation which interpolates between the two asymptotics.

Although the proof that the correlator of twist fields satisfies σPVI has not yet been found, there are further confirmations of Doyon's hypothesis: long-distance asymptotics (4.2) with $b=0$ and the exponent zz' in the short-distance power law (7.2) reproduce the conjectures of [10].

The QFT analogy also implies that for real $z,z'\in(0,1)$ such that $0<z+z'<1$ and $w'=w-z-z'$ (this corresponds to $b=0$) the constant C in (7.2) can be expressed in terms of vacuum expectation values of twist fields, which have been computed in [10] (see also [22]). The resulting conjectural evaluation is:

$$C\big|_{w'=w-z-z'}=G\left[\begin{array}{c}1-z,1+z,1-z',1+z',1+w,1+w,1+z+z'+w,1-z-z'+w\\ 1-z-z',1+z+z',1+z+w,1-z+w,1+z'+w,1-z'+w\end{array}\right], \tag{7.4}$$

where $G\left[\begin{array}{c}a_1,\ldots,a_m\\ b_1,\ldots,b_n\end{array}\right]=\dfrac{\prod_{k=1}^m G(a_k)}{\prod_{k=1}^n G(b_k)}$ and $G(x)$ denotes the Barnes function:

$$G(x+1)=(2\pi)^{x/2}\exp\left\{\frac{\psi(1)x^2-x(x+1)}{2}\right\}\prod_{n=1}^\infty\left[\left(1+\frac{x}{n}\right)^n\exp\left\{-x+\frac{x^2}{2n}\right\}\right].$$

In spite of what one might expect, extension of the above approach to the case $b\neq0$ turns out to be rather complicated. However, the simple structure of (7.4) and the symmetries of the ${}_2F_1$ kernel suggest the following:

Conjecture 8. *Under assumptions of Proposition 7, the constant C in the asymptotic expansions (7.2), (7.3) is given by*

$$C=G\left[\begin{array}{c}1-z,1+z,1-z',1+z',1+w,1+w',1+z+z'+w,1+z+z'+w'\\ 1-z-z',1+z+z',1+z+w,1+z+w',1+z'+w,1+z'+w'\end{array}\right]. \tag{7.5}$$

The formula (7.5) is clearly compatible with (7.4) and (S1)–(S2). It has been checked both numerically and analytically as described below.

8. Numerics

To verify Conjecture 8, one can proceed in the following way:

(1) The solution of PVI associated to the ${}_2F_1$ kernel solution $D(t)$ of σPVI (uniquely determined by (3.6), (3.7)) has the following asymptotic behavior as $t \to 0$:

$$q(t) = t - \lambda_0\, t^{1+z+z'+w+w'} + O(t^{2+z+z'+w+w'}), \tag{8.1}$$

$$\lambda_0 = \frac{(1+z+z'+w+w')^2}{(z+w)(z'+w)}\,\kappa.$$

(2) In fact one can show that in this case

$$q(t) = t - \lambda_0\, t^{1+z+z'+w+w'}(1-t)^{1+z-z'}\,{}_2F_1\left[\begin{array}{c} z+w,\, 1+z+w' \\ 1+z+z'+w+w' \end{array}\bigg|\, t\right]^2 + O(t^{2+2(z+z'+w+w')}). \tag{8.2}$$

(3) Use this asymptotics as initial condition and integrate the corresponding PVI equation numerically for some admissible choice of $\boldsymbol{\theta}$. It is then instructive to check Proposition 7 by verifying that for $0 < \mathrm{Re}(z+z') < 1$ the asymptotic expansion of $q(t)$ as $t \to 1$ is given by

$$q(t) = 1 - \lambda_1\,(1-t)^{1-z-z'} + o\left((1-t)^{1-\mathrm{Re}(z+z')}\right),$$

where

$$\lambda_1 = \Gamma\left[\begin{array}{c} z+z', z+z', 1-z, 1-z', w, 1+w' \\ 1-z-z', 1-z-z', z, z', z+z'+w, 1+z+z'+w' \end{array}\right] = \frac{(1-z-z')^2}{w\,(z+z'+w')}\, a^- .$$

Similarly, for $z+z'=0$ one has a logarithmic behavior,

$$q(t) = 1 + (1-t)\left[z^2\left(\tilde{\Omega} + w^{-1} - 1\right)^2 - 1\right] + O\left((1-t)^2\ln^4(1-t)\right).$$

(4) Finally, use $q(t)$ and the initial condition $D(t) \sim 1$ as $t \to 0$ to compute $D(t)$ from (2.7), (2.9). Looking at the asymptotics of $D(t)$ as $t \to 1$, one can numerically check the formula (7.5) for C.

9. Special solutions check

For special choices of parameters and initial conditions Painlevé VI equation can be solved explicitly. All explicit solutions found so far are either algebraic or of Picard or Riccati type. Algebraic solutions have been classified in [21]; up to parameter equivalence, their list consists of 3 continuous families and 45 exceptional solutions.

It turns out that the parameters of exceptional algebraic solutions cannot be transformed to satisfy ${}_2F_1$ kernel constraints $p_0 = p_{0t}$, $p_t = 2$. Continuous families, however, do contain representatives verifying these conditions. Explicit computation of the corresponding τ-functions provides a number of analytic tests of Conjecture 8, some of which are presented below. Our notation for PVI Bäcklund transformations follows Table 1 in [21].

Example 9. Painlevé VI equation with parameters $\boldsymbol{\theta} = (1, \theta_1, 0, \theta_1)$ is satisfied by

$$q(t) = 1 - \frac{(2\theta_1 - 1) - (2\theta_1+1)\sqrt{1-t}}{(2\theta_1-3) - (2\theta_1-1)\sqrt{1-t}}\,\sqrt{1-t}.$$

This two-branch solution is obtained by applying Bäcklund transformation $s_\delta s_x s_y s_z s_\delta s_z s_\delta P_{xy}$ to Solution II in [21] (set $\theta_a = 1$, $\theta_b = \theta_1$). An explicit formula for the corresponding JMU τ-function can be found from (2.7), (2.9):

$$\tau_{JMU}(t) = \left[\frac{2\,(1-t)^{1/4}}{1+\sqrt{1-t}}\right]^{\frac{1-4\theta_1^2}{4}}.$$

Note that $\tau_{JMU}(t\to 0) = 1 - \frac{1-4\theta_1^2}{128}\,t^2 + O\left(t^3\right)$, and therefore $\tau_{JMU}(t)$ coincides with the hypergeometric kernel determinant $D(t)$ if we set $z = w = \frac{1+2\theta_1}{4}$, $z' = w' = \frac{1-2\theta_1}{4}$.

The asymptotics of $\tau_{JMU}(t)$ as $t \to 1$ has the form

$$\tau_{JMU}(t) = 2^{\frac{1-4\theta_1^2}{4}}\,(1-t)^{\frac{1-4\theta_1^2}{16}}\left(1 + O\left(\sqrt{1-t}\right)\right),$$

which implies that $C = 2^{\frac{1-4\theta_1^2}{4}}$. To verify that this coincides with the expression

$$C = G\left[\begin{array}{c} \frac{3+2\theta_1}{4}, \frac{3-2\theta_1}{4}, \frac{5+2\theta_1}{4}, \frac{5+2\theta_1}{4}, \frac{5-2\theta_1}{4}, \frac{5-2\theta_1}{4}, \frac{7+2\theta_1}{4}, \frac{7-2\theta_1}{4}, \\ \frac12, \frac32, \frac32, \frac32, \frac{3+2\theta_1}{2}, \frac{3-2\theta_1}{2} \end{array}\right].$$

given by Conjecture 8, one can use the recursion relation $G(z+1) = \Gamma(z)G(z)$, the duplication formulas (A.1), (A.3) for Barnes and gamma functions, and the value of $G\left(\frac12\right)$ from Appendix A.

Example 10. Consider the rational curve

$$q = \frac{(s+1)(s-2)(5s^2+4)}{s(s-1)(5s^2-4)}, \qquad t = \frac{(s+1)^2(s-2)}{(s-1)^2(s+2)}.$$

It defines a three-branch solution of PVI with parameters $\boldsymbol{\theta} = (2, 0, 0, 2/3)$, which can be obtained from Solution III in [21] (with $\theta = 0$) by the transformation $t_x = s_x s_\delta \left(s_y s_z s_\infty s_\delta\right)^2$.

The associated τ-function is given by

$$\tau_{JMU}(t(s)) = \frac{3^{\frac{15}{8}}}{2^{\frac{25}{9}}}\cdot\frac{s\,(s+2)^{\frac89}}{(s+1)^{\frac{15}{8}}(s-1)^{\frac{7}{72}}},$$

where the normalization constant is introduced for convenience. The map $t(s)$ bijectively maps the interval $(2, \infty)$ onto $(0, 1)$. Choosing the corresponding solution branch one finds that

$$\tau_{JMU}(t\to 0) = 1 - \frac{16}{19683}\,t^3 + O\left(t^4\right),$$
$$\tau_{JMU}(t\to 1) \sim 3^{\frac{15}{8}}\cdot 2^{-\frac{17}{6}}\cdot(1-t)^{\frac{1}{36}}.$$

First asymptotics implies that $\tau_{JMU}(t)$ coincides with $D(t)$ provided $z = z' = \frac16$, $w = \frac76$, $w' = \frac12$. From the second asymptotics we obtain $C = 3^{\frac{15}{8}}\cdot 2^{-\frac{17}{6}}$, whereas Conjecture 8 gives

$$C = G\left[\begin{array}{c} \frac32, \frac52, \frac56, \frac56, \frac76, \frac76, \frac{11}{6}, \frac{13}{6} \\ \frac23, \frac43, \frac53, \frac53, \frac73, \frac73 \end{array}\right].$$

Equality of both expressions can be shown using the known evaluations of $G\left(\frac{k}{6}\right)$, $k = 1\ldots 5$, see [1] or Appendix A.

Example 11. Applying the transformation $\left(s_\delta s_x s_y\right)^3 s_z s_\infty s_\delta r_x$ to Solution IV in [21] and setting $\theta = 0$, one obtains a four-branch solution of PVI with $\boldsymbol{\theta} = (1, 1/2, 0, 1)$ parameterized by

$$q = \frac{s(2-s)(5s^2-15s+12)}{(3-s)(3-2s)}, \qquad t = \frac{s(2-s)^3}{3-2s}.$$

The corresponding τ-function has the form

$$\tau_{JMU}(t(s)) = \frac{2^{\frac{5}{12}}}{3^{\frac{15}{16}}}\cdot\frac{(3-s)^{\frac{15}{16}}}{(2-s)^{\frac{5}{12}}(1-s)^{\frac{5}{48}}}.$$

Choose the solution branch with $s \in (0,1)$. From the asymptotics $\tau_{JMU}(t \to 0) = 1 + \frac{15}{2048} t^2 + O\left(t^3\right)$ follows that $\tau_{JMU}(t)$ coincides with $D(t)$ provided $z = \frac{5}{12}$, $z' = -\frac{1}{12}$, $w = \frac{5}{6}$, $w' = -\frac{1}{6}$. Leading term in the asymptotic behavior of $\tau_{JMU}(t)$ as $t \to 1$ is

$$\tau_{JMU}(t \to 1) \sim 2^{\frac{25}{18}} \cdot 3^{-\frac{15}{16}} \cdot (1-t)^{-\frac{5}{144}},$$

so that we have $C = 2^{\frac{25}{18}} \cdot 3^{-\frac{15}{16}}$. On the other hand, Conjecture 8 implies that

$$C = G \left[\begin{array}{c} \frac{5}{6}, \frac{7}{6}, \frac{11}{6}, \frac{13}{6}, \frac{7}{12}, \frac{11}{12}, \frac{13}{12}, \frac{17}{12} \\ \frac{2}{3}, \frac{4}{3}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4} \end{array} \right].$$

To prove that these expressions are equivalent, (i) use the multiplication formula (A.1) with $n=2$ and $z = \frac{1}{12}, \frac{5}{12}$ to compute $G\left(\frac{1}{12}\right) G\left(\frac{5}{12}\right) G\left(\frac{7}{12}\right) G\left(\frac{11}{12}\right)$ and (ii) combine the resulting expression with the evaluations of $G\left(\frac{k}{4}\right)$, $G\left(\frac{k}{6}\right)$.

10. Limiting kernels

10.1. **Flat space limit: PVI → PV.** The interpretation of $D(t)$ as a determinant of a Dirac operator (Section 4) suggests to consider the flat space limit $R \to \infty$. This corresponds to the following scaling limit of the ${}_2F_1$ kernel:

$$w' \to +\infty, \qquad 1 - t \sim \frac{s}{w'}, \qquad s \in (0, \infty).$$

In this limit, $D(t)$ transforms into the Fredholm determinant $D_L(s) = \det\left(1 - K_L\big|_{(s,\infty)}\right)$ with the kernel

$$K_L(x,y) = \lim_{w' \to +\infty} \frac{1}{w'} K\left(1 - \frac{x}{w'}, 1 - \frac{y}{w'}\right) = \lambda_L \frac{A_L(x) B_L(y) - B_L(x) A_L(y)}{x - y},$$

$$A_L(x) = x^{-\frac{1}{2}} W_{\frac{1}{2} - \frac{z+z'+2w}{2}, \frac{z-z'}{2}}(x), \qquad B_L(x) = x^{-\frac{1}{2}} W_{-\frac{1}{2} - \frac{z+z'+2w}{2}, \frac{z-z'}{2}}(x),$$

$$\lambda_L = \frac{\sin \pi z \sin \pi z'}{\pi^2} \Gamma\left[1 + z + w, 1 + z' + w\right],$$

where $W_{\alpha,\beta}(x)$ denotes the Whittaker's function of the 2nd kind. $K_L(x,y)$ is the so-called Whittaker kernel (see e.g. [4]), which plays the same role in the harmonic analysis on the infinite symmetric group as the ${}_2F_1$ kernel does for $U(\infty)$.

The function $\sigma_L(s) = s \frac{d}{ds} \ln D_L(s)$ satisfies a Painlevé V equation written in σ-form:

$$(10.1) \quad \left(s \sigma_L''\right)^2 = \left(2 \left(\sigma_L'\right)^2 - (z + z' + 2w + s)\sigma_L' + \sigma_L\right)^2 - 4\left(\sigma_L'\right)^2 \left(\sigma_L' - z - w\right)\left(\sigma_L' - z' - w\right).$$

This can be shown by considering the appropriate limit of the σPVI equation for $D(t)$. An initial condition for (10.1) is provided by the asymptotics

$$D_L(s \to \infty) = 1 - \lambda_L e^{-s} s^{-z-z'-2w-2} \left(1 + O\left(s^{-1}\right)\right).$$

To link our notation with the one used in the PV part of Jimbo's paper [16], we should set $(\theta_0, \theta_t, \theta_\infty)^{(V)}_{\text{Jimbo}} = (z' + w, -z - w, z - z')$, which gives $D_L(s) = e^{\frac{(z+w)s}{2}} \tau^{(V)}_{\text{Jimbo}}(s)$. This in turn allows to obtain from Theorem 3.1 in [16] the asymptotics of $D_L(s)$ as $s \to 0$:

Proposition 12. *Assume that* $0 \leq \operatorname{Re}(z + z') < 1$ *and* $z, z', w, z + z' + w \notin \mathbb{Z}$.

(1) *If* $z + z' \neq 0$, *then*

$$D_L(s) = C_L s^{zz'} \left[1 + \frac{zz'(z + z' + 2w)}{(z + z')^2} s - a_L^+ s^{1+z+z'} - a_L^- s^{1-z-z'} + O\left(s^{2 - 2\operatorname{Re}(z+z')}\right)\right],$$

with

$$a_L^\pm = \Gamma \left[\begin{array}{c} \mp z \mp z', \mp z \mp z', 1 \pm z, 1 \pm z', 1 + w + \frac{z+z'}{2} \pm \frac{z+z'}{2} \\ 2 \pm z \pm z', 2 \pm z \pm z', \mp z, \mp z', w + \frac{z+z'}{2} \mp \frac{z+z'}{2} \end{array} \right].$$

(2) *If* $z+z'=0$, *then*

$$D_L(s) = C_L s^{-z^2}\left[1 + z^2 ws\big(\tilde{\Omega}_L^2 + 2\tilde{\Omega}_L + 3\big) + z^2 s\big(\tilde{\Omega}_L + 1\big) + o(s)\right],$$

where $\tilde{\Omega}_L = 1 - a'_L - \ln s$ *and* $a'_L = \psi(1+z) + \psi(1-z) + \psi(1+w) - 4\psi(1)$.

Note that the same result is obtained by considering the formal limit of the leading terms in the asymptotics of $D(t)$. This further suggests an expression for constant C_L:

Conjecture 13. *Under assumptions of Proposition 12, we have*

$$C_L = \lim_{w'\to\infty} (w')^{-zz'} C = G\left[\begin{array}{c} 1-z, 1+z, 1-z', 1+z', 1+w, 1+z+z'+w \\ 1-z-z', 1+z+z', 1+z+w, 1+z'+w \end{array}\right].$$

10.2. **Zero field limit: PV → PIII.** Next we consider the limit of vanishing magnetic field, $B\to 0$. In terms of the parameters of the Whittaker kernel, this translates into

$$w\to+\infty, \qquad s\sim\frac{\xi}{w}, \qquad \xi\in(0,\infty).$$

The scaled kernel is given by

$$K_M(x,y) = \lim_{w\to+\infty}\frac{1}{w}K_L\left(\frac{x}{w},\frac{y}{w}\right) = \frac{\sin\pi z\sin\pi z'}{\pi^2}\cdot\frac{A_M(x)B_M(y)-B_M(x)A_M(y)}{x-y},$$

$$A_M(x) = 2\sqrt{x}\,K_{z'-z+1}\left(2\sqrt{x}\right), \qquad B_M(x) = 2\,K_{z'-z}\left(2\sqrt{x}\right),$$

where $K_\alpha(x)$ is the Macdonald function.

Denote $D_M(\xi) = \det\left(1 - K_M\big|_{(s,\infty)}\right)$ and introduce $\sigma_M(\xi) = \xi\frac{d}{d\xi}\ln D_M(\xi)$. Then $\sigma_M(\xi)$ solves the σ-version of a particular Painlevé III equation:

$$\left(\xi\sigma''_M\right)^2 = 4\sigma'_M(\sigma'_M-1)(\sigma_M-\xi\sigma'_M) + (z-z')^2\left(\sigma'_M\right)^2. \tag{10.2}$$

To match the notation in [16], we have to set $(\theta_0,\theta_\infty)^{(III)}_{\text{Jimbo}} = (z'-z, z-z')$, which gives $D_L(s) = e^{\xi}\tau^{(III)}_{\text{Jimbo}}(\xi)$. The appropriate initial condition for this σPIII is given by

$$D_M(\xi\to\infty) = 1 - \frac{\sin\pi z\sin\pi z'}{4\pi}\cdot\frac{e^{-4\sqrt{\xi}}}{\sqrt{\xi}}\left(1 + \frac{4(z-z')^2-3}{8\sqrt{\xi}} + O\left(\xi^{-1}\right)\right). \tag{10.3}$$

The asymptotics of $D_M(\xi)$ as $\xi\to 0$ can now be obtained from Theorem 3.2 in [16]:

Proposition 14. *Assume that* $0\le\operatorname{Re}(z+z')<1$ *and* $z,z'\notin\mathbb{Z}$.

(1) *If* $z+z'\neq 0$, *then*

$$D_M(\xi\to 0) = C_M\,\xi^{zz'}\left[1 + \frac{2zz'}{(z+z')^2}\xi - a_M^+\xi^{1+z+z'} - a_M^-\xi^{1-z-z'} + O\left(\xi^{2-2\operatorname{Re}(z+z')}\right)\right],$$

with $a_M^\pm = \Gamma\left[\begin{array}{c} \mp z\mp z', \mp z\mp z', 1\pm z, 1\pm z' \\ 2\pm z\pm z', 2\pm z\pm z', \mp z, \mp z' \end{array}\right]$.

(2) *If* $z+z'=0$, *then*

$$D_M(\xi\to 0) = C_M\,\xi^{-z^2}\left[1 + z^2\xi\big(\tilde{\Omega}_M^2 + 2\tilde{\Omega}_M + 3\big) + o(\xi)\right],$$

where $\tilde{\Omega}_M = 1 - a'_M - \ln\xi$ *and* $a'_M = \psi(1+z) + \psi(1-z) - 4\psi(1)$.

Analogously to the above, we suggest a conjectural expression for C_M:

Conjecture 15. *Under assumptions of Proposition 14, we have*

$$C_M = \lim_{w\to\infty} w^{-zz'}C_L = G\left[\begin{array}{c} 1-z, 1+z, 1-z', 1+z' \\ 1-z-z', 1+z+z' \end{array}\right].$$

Partial proof. This formula can in fact be proved for real $z = z' \in \left[0, \frac{1}{2}\right)$, though in an indirect way. Consider the solution $\psi(r)$ of the radial sinh-Gordon equation

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = \frac{1}{2}\sinh 2\psi,$$

satisfying the boundary condition $\psi(r,\nu) \sim 2\nu K_0(r)$ as $r \to +\infty$. Define the function

$$\tau(r,\nu) = \exp\left\{\frac{1}{2}\int_r^\infty u\left[\sinh^2\psi(u,\nu) - \left(\frac{d\psi}{du}\right)^2\right]du\right\}.$$

and consider the logarithmic derivative $\tilde{\sigma}(\xi) = \xi\frac{d}{d\xi}\ln\tau(2\sqrt{\xi},\nu)$. It is straightforward to show that $\tilde{\sigma}(\xi)$ satisfies σPIII equation (10.2) with $z = z'$. Further, a little calculation shows that, as $r \to +\infty$,

$$\tau(r,\nu) = 1 - \pi\nu^2\,\frac{e^{-2r}}{2r}\left(1 - \frac{3}{4r} + O\left(r^{-2}\right)\right).$$

Comparing this asymptotics with (10.3), we conclude that $D_M(\xi)\Big|_{z=z'} = \tau\Big(2\sqrt{\xi}, \pm\frac{\sin\pi z}{\pi}\Big)$.

On the other hand, $\tau(r,\nu) = \tau_B^{-1}(r,\nu)$, where $\tau_B(r,\nu)$ is a special case of the bosonic 2-point tau function of Sato, Miwa and Jimbo, which can be represented as an infinite series of integrals (formulas (4.5.30)–(4.5.31) in [24] with $l_1 = l_2$). By direct asymptotic analysis of this series, Tracy [25] has proved that for $\nu \in \left[0, \frac{1}{\pi}\right)$ it has the following behavior as $r \to 0$:

$$\tau_B(r,\nu) = e^{\beta(\nu)} r^{-\alpha(\nu)}\left(1 + o(1)\right),$$

with

$$\alpha(\nu) = \frac{\sigma^2(\nu)}{2}, \qquad \sigma(\nu) = \frac{2}{\pi}\arcsin\pi\nu,$$

$$\beta(\nu) = 3\alpha(\nu)\ln 2 + \frac{1}{2}\ln(1-\pi^2\nu^2) - 2\ln\cos\frac{\pi\sigma(\nu)}{2} - 2\ln\left(G\left[\begin{array}{cc}\frac{1}{2}, & \frac{1}{2}\\ \frac{1+\sigma(\nu)}{2}, & \frac{1-\sigma(\nu)}{2}\end{array}\right]\right).$$

From $\nu = \pm\frac{\sin\pi z}{\pi}$ one readily obtains $\sigma^2 = 2\alpha = 4z^2$. Thus, in order to show that $\beta(\nu)$ reproduces the conjectured expression for C_M with $z = z'$, it is sufficient to prove the identity

$$G\left[\begin{array}{c}1+z, 1+z, 1-z, 1-z\\ 1+2z, 1-2z\end{array}\right] = 2^{-4z^2}\cos\pi z\; G\left[\begin{array}{c}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ \frac{1}{2}+z, \frac{1}{2}+z, \frac{1}{2}-z, \frac{1}{2}-z\end{array}\right].$$

This, however, is a simple consequence of the duplication formula for Barnes function and the known evaluation of $G\left(\frac{1}{2}\right)$. □

Appendix A

Multiplication formula for Barnes function [28]:

(A.1) $$\ln G(nx) = \left(\frac{n^2x^2}{2} - nx\right)\ln 2 - \frac{(n-1)(nx-1)}{2}\ln 2\pi + \frac{5}{12}\ln n - \frac{n^2-1}{12} + \left(n^2-1\right)\ln A + \sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\ln G\left(x + \frac{j+k}{n}\right),$$

where $A = \exp\left(\frac{1}{12} - \zeta'(-1)\right)$ denotes Glaisher's constant.

Asymptotic expansion as $|z| \to \infty$, $\arg z \neq \pi$:

(A.2) $$\ln G(1+z) = \left(\frac{z^2}{2} - \frac{1}{12}\right)\ln z - \frac{3z^2}{4} + \frac{z}{2}\ln 2\pi - \ln A + \frac{1}{12} + O\left(\frac{1}{z^2}\right).$$

Special values (see, e.g. [1]):

$$\ln G\left(\frac{1}{2}\right) = \frac{\ln 2}{24} - \frac{\ln\pi}{4} - \frac{3}{2}\ln A + \frac{1}{8},$$
$$\ln G\left(\frac{1}{3}\right) = \frac{\ln 3}{72} + \frac{\pi}{18\sqrt{3}} - \frac{2}{3}\ln\Gamma\left(\frac{1}{3}\right) - \frac{4}{3}\ln A - \frac{1}{12\pi\sqrt{3}}\,\psi'\left(\frac{1}{3}\right) + \frac{1}{9},$$
$$\ln G\left(\frac{2}{3}\right) = \frac{\ln 3}{72} + \frac{\pi}{18\sqrt{3}} - \frac{1}{3}\ln\Gamma\left(\frac{2}{3}\right) - \frac{4}{3}\ln A - \frac{1}{12\pi\sqrt{3}}\,\psi'\left(\frac{2}{3}\right) + \frac{1}{9},$$
$$\ln G\left(\frac{1}{6}\right) = -\frac{\ln 12}{144} + \frac{\pi}{20\sqrt{3}} - \frac{5}{6}\ln\Gamma\left(\frac{1}{6}\right) - \frac{5}{6}\ln A - \frac{1}{40\pi\sqrt{3}}\,\psi'\left(\frac{1}{6}\right) + \frac{5}{72},$$
$$\ln G\left(\frac{5}{6}\right) = -\frac{\ln 12}{144} + \frac{\pi}{20\sqrt{3}} - \frac{1}{6}\ln\Gamma\left(\frac{5}{6}\right) - \frac{5}{6}\ln A - \frac{1}{40\pi\sqrt{3}}\,\psi'\left(\frac{5}{6}\right) + \frac{5}{72},$$
$$\ln G\left(\frac{1}{4}\right) = -\frac{3}{4}\ln\Gamma\left(\frac{1}{4}\right) - \frac{9}{8}\ln A + \frac{3}{32} - \frac{K}{4\pi},$$
$$\ln G\left(\frac{3}{4}\right) = -\frac{1}{4}\ln\Gamma\left(\frac{3}{4}\right) - \frac{9}{8}\ln A + \frac{3}{32} + \frac{K}{4\pi},$$

where K is Catalan's constant.

When checking Conjecture 8 with explicit examples, one also needs the relations

$$\Gamma(nx) = (2\pi)^{-\frac{n-1}{2}} n^{nx-\frac{1}{2}} \prod_{k=0}^{n-1}\Gamma\left(x+\frac{k}{n}\right), \qquad \psi'(x)+\psi'(1-x) = \frac{\pi^2}{\sin^2\pi x}. \tag{A.3}$$

References

[1] V. S. Adamchik, *On the Barnes function*, Proc. 2001 Int. Symp. Symbolic and Algebraic Computation, Academic Press, (2001), 15-20.

[2] M. Adler, T. Shiota, P. van Moerbeke, *Random matrices, vertex operators and the Virasoro algebra*, Phys. Lett. **A208**, (1995), 67–78.

[3] J. Baik, R. Buckingham, J. DiFranco, *Asymptotics of Tracy-Widom distributions and the total integral of a Painlevé II function*, Comm. Math. Phys. **280**, (2008), 463–497; preprint `arXiv:0704.3636 [math.FA]`.

[4] A. Borodin, *Harmonic analysis on the infinite symmetric group and the Whittaker kernel*, St. Petersburg Math. J. **12**, (2001), 733–759.

[5] A. Borodin, G. Olshanski, *Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes*, Ann. Math. **161**, (2005), 1319–1422; preprint `math/0109194 [math.RT]`.

[6] A. Borodin, P. Deift, *Fredholm determinants, Jimbo-Miwa-Ueno tau-functions, and representation theory*, Comm. Pure Appl. Math. **55**, (2002), 1160–1230; preprint `math-ph/0111007`.

[7] O. Costin, R. D. Costin, *Asymptotic properties of a family of solutions of the Painlevé equation P_{VI}*, Int. Math. Res. Notices **22**, (2002), 1167-1182; preprint `math/0202235 [math.CA]`.

[8] P. Deift, A. Its, I. Krasovsky, *Asymptotics of the Airy-kernel determinant*, Comm. Math. Phys. **278**, (2008), 643–678; preprint `math/0609451 [math.FA]`.

[9] P. Deift, A. Its, I. Krasovsky, X. Zhou, *The Widom-Dyson constant for the gap probability in random matrix theory*, J. Comput. Appl. Math. **202**, (2007), 26–47; preprint `math/0601535 [math.FA]`.

[10] B. Doyon, *Two-point correlation functions of scaling fields in the Dirac theory on the Poincaré disk*, Nucl. Phys. **B675**, (2003), 607–630; preprint `hep-th/0304190`.

[11] B. Dubrovin, M. Mazzocco, *Monodromy of certain Painlevé VI transcendents and reflection groups*, Inv. Math. **141**, (2000), 55–147; preprint `math.AG/9806056`.

[12] F. J. Dyson, *Fredholm determinants and inverse scattering problems*, Comm. Math. Phys. **47**, (1976), 171–183.

[13] T. Ehrhardt, *Dyson's constant in the asymptotics of the Fredholm determinant of the sine kernel*, Comm. Math. Phys. **262**, (2006), 317–341; preprint `math/0401205 [math.FA]`.

[14] L. Haine, J.-P. Semengue, *The Jacobi polynomial ensemble and the Painlevé VI equation*, J. Math. Phys. **40**, (1999), 2117–2134.

[15] A. R. Its, A. G. Izergin, V. E. Korepin, N. A. Slavnov, *Differential equations for quantum correlation functions*, Int. J. Mod. Phys. **B4**, (1990), 1003–1037.

[16] M. Jimbo, *Monodromy problem and the boundary condition for some Painlevé equations*, Publ. RIMS, Kyoto Univ. **18**, (1982), 1137–1161.

[17] M. Jimbo, T. Miwa, Y. Môri, M. Sato, *Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent*, Physica **1D**, (1980), 80–158.

[18] M. Jimbo, T. Miwa, K. Ueno, *Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I*, Physica **2D**, (1981), 306–352.

[19] I. V. Krasovsky, *Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on an arc of the unit circle*, Int. Math. Res. Not. **2004**, (2004), 1249–1272; preprint `math/0401258 [math.FA]`.

[20] O. Lisovyy, *On Painlevé VI transcendents related to the Dirac operator on the hyperbolic disk*, J. Math. Phys. **49**, (2008), 093507; preprint `arXiv:0710.5744 [math-ph]`.

[21] O. Lisovyy, Yu. Tykhyy, *Algebraic solutions of the sixth Painlevé equation*, preprint `arXiv:0809.4873 [math.CA]`.

[22] S. Lukyanov, A. B. Zamolodchikov, *Exact expectation values of local fields in quantum sine-Gordon model*, Nucl. Phys. **B493**, (1997), 571–587; preprint `hep-th/9611238`.

[23] J. Palmer, M. Beatty, C. A. Tracy, *Tau functions for the Dirac operator on the Poincaré disk*, Comm. Math. Phys. **165**, (1994), 97–173; preprint `hep-th/9309017`.

[24] M. Sato, T. Miwa, M. Jimbo, *Holonomic quantum fields IV*, Publ. RIMS, Kyoto Univ. **15**, (1979), 871–972.

[25] C. A. Tracy, *Asymptotics of a τ-function arising in the two-dimensional Ising model*, Comm. Math. Phys. **142**, (1991), 297–311.

[26] C. A. Tracy, H. Widom, *Level-spacing distributions and the Airy kernel*, Comm. Math. Phys. **159**, (1994), 151–174; preprint `hep-th/9211141`.

[27] C. A. Tracy, H. Widom, *Fredholm determinants, differential equations and matrix models*, Comm. Math. Phys. **163**, (1994), 33–72 ; preprint `hep-th/9306042`.

[28] I. Vardi, *Determinants of Laplacians and multiple gamma functions*, SIAM J. Math. Anal. **19**, (1988), 493-507.

[29] T. T. Wu, B. M. McCoy, C. A. Tracy, E. Barouch, *Spin-spin correlation functions for the two-dimensional Ising model: exact theory in the scaling region*, Phys. Rev. **B13**, (1976), 316–374.

Laboratoire de Mathématiques et Physique Théorique CNRS/UMR 6083, Université de Tours, Parc de Grandmont, 37200 Tours, France

E-mail address: `lisovyi@lmpt.univ-tours.fr`

Bogolyubov Institute for Theoretical Physics, 03680 Kyiv, Ukraine