Painlevé functions and conformal blocks

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Abstract. We outline recent developments relating Painlevé equations and 2D conformal field theory. Generic tau functions of Painlevé VI and Painlevé III_3 are written as linear combinations of $c = 1$ conformal blocks and their irregular limits. This provides explicit combinatorial series representations of the tau functions, and helps to establish connection formula for the tau function in the Painlevé VI case.

1. Introduction

Painlevé equations [4, 5, 10] have long been a major research subject. After their discovery at the turn of 20th century, a new surge of interest emerged in the seventies owing to the development of the inverse scattering method and various applications in integrable models and random matrix theory.

The important progress achieved since then in the study of geometric and analytic properties of Painlevé equations is mainly due to their intimate relation to monodromy preserving deformations of systems of linear ODEs with rational coefficients. It is now known that Painlevé transcendents share many properties of the classical special functions, including reasonably simple transformation properties and connection formulas. The main difference is the absence of explicit series/integral representations of Painlevé functions and their lacking representation-theoretic interpretation.

However, it was recently observed in [13, 14] that the tau functions of Painlevé VI, V and III can be simply expressed in terms of conformal blocks of the Virasoro algebra with the central charge $c = 1$ and their irregular limits. These observations appear even more intriguing in the light of AGT correspondence [2] which provides explicit combinatorial series representations for conformal blocks and, consequently, complete critical expansions of the corresponding Painlevé functions.

The results of [13, 14] beg for conceptual explanation. The present paper attempts to initiate the study of algebraic stuctures behind Painlevé equations in the hope that their understanding will eventually lead to a rigorous proof of conformal expansions. Focusing on the simplest Painlevé III_3 case, we will show that these expansions are equivalent to certain bilinear differential identities for conformal blocks (given by the squares of Whittaker vectors in the case at hand).

Our second aim is to illustrate the use of conformal expansions by finding a connection formula for generic Painlevé VI tau function. Although connection problem for classical Painlevé VI transcendents was essentially solved in [17], its tau function version involves their quadratures and therefore looks much more complicated. In the context of random matrix theory, connection coefficients of this type arise in the study of large gap asymptotics for integrable kernels and are called Dyson constants, see e.g. [6, 7, 9, 20].

The organization of the paper is as follows. In Section 2, we recall how conformal blocks arise in 2D CFT and explain two ways of their computation: direct approach based on the inversion of the Kac-Shapovalov matrix, and AGT combinatorial representation. We also briefly outline the algebraic setting leading to the notion of irregular conformal blocks. In Section 3, we recall Jimbo asymptotic formula, expressing the critical asymptotics of Painlevé VI tau function in terms of monodromy data, and describe its all-order analog using $c = 1$ Virasoro conformal blocks. Section 4 explains how the proof of such critical expansions can be reduced to showing certain bilinear identities satisfied by conformal block functions. Finally, Section 5 contains a sketch of the derivation of connection coefficient for generic Painlevé VI tau function.

2. Computation of conformal blocks

2.1. CFT basics

The main objective of 2D conformal field theory [3, 8, 27] is the calculation of correlation functions of local fields, and one of its nicest features is that the correlators are almost completely fixed by the underlying symmetry.

The space of local fields contains a (possibly infinite and uncountable) set of chiral primary fields $\mathcal{O}$ characterized by their conformal dimensions $\Delta \in \mathbb{C}$. Such fields transform as holomorphic forms of weight Δ under conformal transformations. In addition to primaries, the theory also contains their descendants obtained by successive operator product expansions (OPE) with a specific field T called the energy-momentum tensor. In this paper, we will be exclusively interested in the fields living on the Riemann sphere $\mathbb{P}^1$.

In the algebraic formulation of CFT, local fields correspond to states in the highest weight representations of the Virasoro algebra

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12}\, n\left(n^2-1\right)\delta_{n+m,0}, \qquad m,n\in\mathbb{Z}. \tag{2.1}$$

One assigns to every primary field $\mathcal{O}$ a highest weight vector $|\Delta\rangle$ and its dual $\langle\Delta|$ such that $L_0|\Delta\rangle = \Delta|\Delta\rangle$, $\langle\Delta|L_0 = \langle\Delta|\Delta$ and $L_n|\Delta\rangle = 0$, $\langle\Delta|L_{-n} = 0$ for any $n > 0$. It is also common to choose an orthonormal basis of primaries so that $\langle\Delta_j|\Delta_k\rangle = \delta_{jk}$. The descendant fields $L_{-\lambda}\mathcal{O}$ are associated to the states $L_{-\lambda}|\Delta\rangle = L_{-\lambda_N}\dots L_{-\lambda_1}|\Delta\rangle$ obtained by the action of lowering operators. As one can always achieve the ordering $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N > 0$ using the commutation relations (2.1), the descendant states may be indexed by partitions identified with Young diagrams. Let $\mathbb{Y}$ denote the set of all partitions. The size $|\lambda| = \sum_{k=1}^N \lambda_k$ of $\lambda \in \mathbb{Y}$ is called the level of descendant $L_{-\lambda}\mathcal{O}$.

The computation of correlation functions is based on the systematic use of OPEs and the possibility to express correlation functions involving descendant fields in terms of correlators of the corresponding primaries. For instance, the OPE of two primary fields $\mathcal{O}_1$ and $\mathcal{O}_2$ has the form

$$\mathcal{O}_2(t)\mathcal{O}_1(0) = \sum_{\alpha} C_{\alpha 21} \sum_{\lambda\in\mathbb{Y}} \beta_\lambda\left(\Delta_\alpha, \Delta_2, \Delta_1\right) t^{\Delta_\alpha - \Delta_1 - \Delta_2 + |\lambda|} L_{-\lambda}\mathcal{O}_\alpha\left(0\right), \quad (2.2)$$

where the index α labels possible intermediate conformal families and $C_{\alpha 21} = \langle\mathcal{O}_\alpha(\infty)\mathcal{O}_2(1)\mathcal{O}_1(0)\rangle$ denotes the three-point function of $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_\alpha$.

The coefficients $\beta_\lambda\left(\Delta_\alpha, \Delta_2, \Delta_1\right)$ of the OPE algebra can be determined by equating correlation functions of both sides of (2.2) with the descendant $L_{-\mu}\mathcal{O}_\alpha(\infty)$:

$$\beta_\lambda\left(\Delta_\alpha, \Delta_2, \Delta_1\right) = \sum_{\mu\in\mathbb{Y}} [Q\left(\Delta_\alpha\right)]^{-1}_{\lambda\mu} \gamma_\mu\left(\Delta_\alpha, \Delta_2, \Delta_1\right). \quad (2.3)$$

Here $Q_{\lambda\mu}\left(\Delta\right)$ is the so-called Kac-Shapovalov matrix involving two-point functions of descendant states,

$$Q_{\lambda\mu}\left(\Delta\right) = \langle\Delta|L_{\lambda_1}\dots L_{\lambda_N} L_{-\mu_M}\dots L_{-\mu_1}|\Delta\rangle, \quad (2.4)$$

where M and N denote the lengths of the partitions μ and λ. The quantity $\gamma_\mu\left(\Delta, \Delta_2, \Delta_1\right)$ is related to three-point function involving one descendant and is known in explicit form:

$$\gamma_\mu\left(\Delta, \Delta_2, \Delta_1\right) = \prod_{j=1}^{M} \Big(\Delta - \Delta_1 + \mu_j\Delta_2 + \sum_{k=1}^{j-1} \mu_k\Big). \quad (2.5)$$

Replacing the products of fields by the OPEs (2.2), one can express any correlator in terms of (non-universal) three-point functions and conformal blocks — the functions which do not depend on a specific model. In the simplest nontrivial case of four-point correlation function

$$\langle\mathcal{O}_4(\infty)\mathcal{O}_3(1)\mathcal{O}_2(t)\mathcal{O}_1(0)\rangle = \sum_{\alpha} C_{43\alpha} C_{\alpha 21} t^{\Delta_\alpha - \Delta_1 - \Delta_2} \mathcal{F}_c\left(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_\alpha; t\right), \quad (2.6)$$

conformal block is explicitly given by

$$\mathcal{F}_c\left(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta; t\right) = \sum_{\lambda,\mu\in\mathbb{Y}} \gamma_\lambda\left(\Delta, \Delta_3, \Delta_4\right) \left[Q\left(\Delta\right)\right]^{-1}_{\lambda\mu} \gamma_\mu\left(\Delta, \Delta_2, \Delta_1\right) t^{|\lambda|}. \quad (2.7)$$

It is often convenient to include the prefactor $t^{\Delta_\alpha - \Delta_1 - \Delta_2}$ in (2.6) into the definition of conformal block. To distinguish between the two conventions, we define

$$\bar{\mathcal{F}}_c\left(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta; t\right) = t^{\Delta - \Delta_1 - \Delta_2} \mathcal{F}_c\left(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta; t\right), \quad (2.8)$$

where the fractional powers here and below will always be understood as defined by the principal branches.

The function (2.7) depends on 6 parameters: four external dimensions $\Delta_{1,2,3,4}$, one intermediate dimension Δ and the central charge c. Although not much is rigorously proven about convergence of the series (2.7) and its analytic behaviour, it is expected that the only branch points of $\mathcal{F}_c$ on the Riemann sphere are 0, 1 and ∞.

2.2. Direct approach

The previously introduced descendant states diagonalize the operator L_0:

$$L_0 L_{-\lambda}|\Delta\rangle = (\Delta + |\lambda|)\, L_{-\lambda}|\Delta\rangle, \qquad \langle\Delta|L_\lambda L_0 = (\Delta + |\lambda|)\, \langle\Delta|L_\lambda.$$

Therefore, Kac-Shapovalov matrix (2.4) is block-diagonal; the only non-vanishing scalar products correspond to descendants from the same level. This allows to compute conformal block function (2.7) order by order using Virasoro commutation relations.

For instance, bringing all $L_{n<0}$ in (2.4) to the left and all $L_{n>0}$ to the right, we easily determine non-zero elements of $Q(\Delta)$ for levels 1 and 2:

$$\begin{cases} Q_{\square\,\square}(\Delta) = 2\Delta, \\ Q_{\square\!\square\,\square\!\square}(\Delta) = 4\Delta + \dfrac{c}{2}, \\ Q_{\square\!\square\,\substack{\square\\\square}}(\Delta) = Q_{\substack{\square\\\square}\,\square\!\square}(\Delta) = 6\Delta, \\ Q_{\substack{\square\\\square}\,\substack{\square\\\square}}(\Delta) = 4\Delta\,(2\Delta + 1)\,. \end{cases} \tag{2.9}$$

Also, from (2.5) we deduce that

$$\begin{cases} \gamma_{\square}(\Delta, \Delta_2, \Delta_1) = \Delta - \Delta_1 + \Delta_2, \\ \gamma_{\square\!\square}(\Delta, \Delta_2, \Delta_1) = \Delta - \Delta_1 + 2\Delta_2, \\ \gamma_{\substack{\square\\\square}}(\Delta, \Delta_2, \Delta_1) = (\Delta - \Delta_1 + \Delta_2)\,(\Delta - \Delta_1 + \Delta_2 + 1)\,. \end{cases} \tag{2.10}$$

Substituting (2.9)–(2.10) into the series (2.7) and computing the inverses of the corresponding blocks of $Q(\Delta)$, one arrives at

$$\begin{aligned} \mathcal{F}_c\,(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta; t) = 1 + \frac{(\Delta - \Delta_1 + \Delta_2)\,(\Delta - \Delta_4 + \Delta_3)}{2\Delta}\, t\, + \\ + \left[\frac{(\Delta - \Delta_1 + \Delta_2)\,(\Delta - \Delta_1 + \Delta_2 + 1)\,(\Delta - \Delta_4 + \Delta_3)\,(\Delta - \Delta_4 + \Delta_3 + 1)}{2\Delta\,(1 + 2\Delta)} + \right. \\ \left. + \frac{(1+2\Delta)\left(\Delta_1 + \Delta_2 + \frac{\Delta(\Delta-1) - 3(\Delta_1 - \Delta_2)^2}{1+2\Delta}\right)\left(\Delta_4 + \Delta_3 + \frac{\Delta(\Delta-1) - 3(\Delta_4 - \Delta_3)^2}{1+2\Delta}\right)}{(1 - 4\Delta)^2 + (c-1)\,(1 + 2\Delta)} \right] \frac{t^2}{2} + O\left(t^3\right). \end{aligned} \tag{2.11}$$

There is no explicit formula expressing generic four-point conformal block in terms of known special functions. However, for some special values of central charge and conformal dimensions $\mathcal{F}_c$ can be computed in a closed form. One of such cases corresponds to the limit $c \to \infty$, in which the Virasoro algebra boils down to $\mathfrak{sl}(2)$-algebra generated by L_0, $L_{\pm 1}$ and conformal block is given by the Gauss hypergeometric function,

$$\mathcal{F}_\infty(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta; t) = {}_2F_1(\Delta - \Delta_1 + \Delta_2, \Delta - \Delta_4 + \Delta_3; 2\Delta; t).$$

Unfortunately, the computation of coefficients of $\mathcal{F}_c$ becomes very cumbersome for large powers of t even with the use of computer algebra. One way to circumvent these difficulties is provided by Zamolodchikov recursive formulas [24, 26]. The next subsection describes yet another approach to this problem.

2.3. AGT representation

It has been recently observed [2] and proved [1] that conformal blocks of 2D CFT coincide with instanton partition functions in 4D $\mathcal{N} = 2$ supersymmetric gauge theories. Mathematically this 2D/4D duality (usually referred to as AGT correspondence) is related to geometric realization of representations of the Virasoro algebra using equivariant cohomology of the moduli spaces of $SU(2)$-instantons.

One of the immediate practical consequences of the AGT relation is an explicit combinatorial representation for $\mathcal{F}_c(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta; t)$. To write it down, it is convenient to parameterize the central charge and conformal dimensions as

$$c = 1 - 6Q^2, \qquad Q = \beta - \beta^{-1}, \qquad \Delta = \frac{c-1}{24} + \sigma^2, \tag{2.12}$$

$$\Delta_k = \frac{c-1}{24} + \theta_k^2, \qquad k = 1, \ldots, 4. \tag{2.13}$$

Then [1, 2] four-point conformal block on $\mathbb{P}^1$ is given by the following double sum over partitions:

$$\mathcal{F}_c(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta; t) = (1-t)^{2(\theta_2 + Q/2)(\theta_3 + Q/2)} \sum_{\lambda, \mu \in \mathbb{Y}} \mathcal{F}_{\lambda,\mu}(\vec{\theta}, \sigma, \beta)\, t^{|\lambda| + |\mu|}, \tag{2.14}$$

$$\mathcal{F}_{\lambda,\mu}(\vec{\theta}, \sigma, \beta) = \prod_{(i,j)\in\lambda} \frac{E_{ij}(\theta_1, \theta_2, \sigma)\, E_{ij}(-\theta_1, \theta_2, \sigma)\, E_{ij}(\theta_4, \theta_3, \sigma)\, E_{ij}(-\theta_4, \theta_3, \sigma)}{F_{ij}(0|\lambda, \lambda)\, F_{ij}(\sigma|\lambda, \mu)} \times \tag{2.15}$$
$$\times \prod_{(i,j)\in\mu} \frac{E_{ij}(\theta_1, \theta_2, -\sigma)\, E_{ij}(-\theta_1, \theta_2, -\sigma)\, E_{ij}(\theta_4, \theta_3, -\sigma)\, E_{ij}(-\theta_4, \theta_3, -\sigma)}{F_{ij}(0|\mu, \mu)\, F_{ij}(-\sigma|\mu, \lambda)}.$$

The products in the last formula are taken over the boxes (i, j) of Young diagrams associated to $\lambda, \mu \in \mathbb{Y}$, the functions E_{ij}, F_{ij} are defined by

$$E_{ij}(\theta, \theta', \sigma) = \theta + \theta' + \sigma + \beta i - \beta^{-1} j - \frac{Q}{2}, \tag{2.16}$$

$$F_{ij}(\sigma|\lambda, \mu) = \left[\beta\left(\lambda'_j - i + \frac{1}{2}\right) + \beta^{-1}\left(\mu_i - j + \frac{1}{2}\right) + 2\sigma\right]^2 - \frac{Q^2}{4}, \tag{2.17}$$

and λ' denotes the partition conjugate to λ. Although the number of bipartitions (λ, μ) of fixed size $N = |\lambda| + |\mu|$ grows rather rapidly ($\sim \frac{\sqrt[4]{3}}{12N^{5/4}} \exp 2\pi\sqrt{\frac{N}{3}}$), the formulas (2.14)–(2.15) provide a very efficient tool of computation of the coefficients of conformal block series.

2.4. Chain vector representation and irregular limit

There exists yet another way to represent conformal block, which follows from (2.7). Consider a sequence of states $|n\rangle$, $n \in \mathbb{Z}$ in the weight representation of the Virasoro algebra defined by the following conditions:

$$|0\rangle = |\Delta\rangle, \qquad |n < 0\rangle = 0, \tag{2.18}$$

$$L_k|n\rangle = \xi_{n,k}|n-k\rangle \qquad \forall k > 0, \tag{2.19}$$

with some coefficients $\xi_{n,k} \in \mathbb{C}$. For the conditions (2.18)–(2.19) to be compatible with the Virasoro commutation relations (2.1), these coefficients should satisfy the relations

$$\xi_{n-k_2,k_1}\xi_{n,k_2} - \xi_{n-k_1,k_2}\xi_{n,k_1} = (k_1 - k_2)\xi_{n,k_1+k_2}.$$

It is straightforward to check that these relations have a two-parameter solution

$$\xi_{n,k} = \begin{cases} \Delta - \Delta_1 + k\Delta_2 + n - k, & k \le n, \\ 0, & k > n. \end{cases}$$

In fact, it can be shown that there are no other solutions. Moreover, if we denote the corresponding states as $|n\rangle_{12}$, then conformal block (2.7) can be written as

$$\mathcal{F}_c\left(\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta; t\right) = \sum_{n=0}^{\infty} {}_{43}\langle n|n\rangle_{21}\, t^n. \tag{2.20}$$

The above construction simplifies in the irregular limit [12]

$$\Delta_{1,4} \to \infty, \qquad t \to \frac{t}{\Delta_1\Delta_4}. \tag{2.21}$$

Rescaling the states as $|n\rangle \to (-\Delta_1)^n|n\rangle$, the conditions (2.18)–(2.19) transform into

$$|0\rangle = |\Delta\rangle, \qquad L_1|n\rangle = |n-1\rangle, \qquad L_2|n\rangle = 0. \tag{2.22}$$

Observe that (2.22) and (2.1) automatically imply that $L_k|n\rangle = 0$ for $k > 2$.

The limit of the conformal block series yields

$$\mathcal{F}_c\left(\Delta; t\right) = \sum_{n=0}^{\infty} \langle n|n\rangle\, t^n = {}_W\langle\Delta|\Delta\rangle_W, \tag{2.23}$$

where the state $|\Delta\rangle_W = \sum_{n=0}^{\infty} t^{\frac{n}{2}}|n\rangle$ is called the Whittaker vector. It has the obvious properties

$$L_1|\Delta\rangle_W = \sqrt{t}\,|\Delta\rangle_W, \qquad L_2|\Delta\rangle_W = 0, \tag{2.24}$$

which are equivalent to the conditions (2.22). Irregular conformal block thus coincides with the norm of this particular state.

Combinatorial representation for $\mathcal{F}_c(\Delta;t)$ can be obtained by taking the appropriate limit of AGT series (2.14)–(2.15). Explicitly,

$$\mathcal{F}_c(\Delta;t) = \sum_{\lambda,\mu\in\mathbb{Y}} \frac{t^{|\lambda|+|\mu|}}{\prod_{(i,j)\in\lambda} F_{ij}(0|\lambda,\lambda)\, F_{ij}(\sigma|\lambda,\mu) \prod_{(i,j)\in\mu} F_{ij}(0|\mu,\mu)\, F_{ij}(-\sigma|\mu,\lambda)}, \tag{2.25}$$

where $F_{ij}(\sigma|\lambda,\mu)$ are defined by the same formulas (2.17).

3. Painlevé VI

3.1. General

The most natural framework for Painlevé equations is the theory of monodromy preserving deformations. Consider, for instance, the rank 2 linear system with four regular singular points on $\mathbb{P}^1$ at $0,t,1,\infty$,

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}\right) Y, \tag{3.1}$$

where the matrices $A_{0,t,1} \in \mathfrak{sl}_2(\mathbb{C})$ are independent of z. The fundamental matrix solution $Y(z)$ is multivalued on $\mathbb{P}^1\backslash\{0,t,1,\infty\}$. The fundamental group $\pi_1(\mathbb{P}^1\backslash\{0,t,1,\infty\})$ is generated by the loops γ_ν ($\nu = 0,t,1,\infty$) shown in Fig. 1, to which we associate monodromy matrices $M_\nu \in SL(2,\mathbb{C})$.

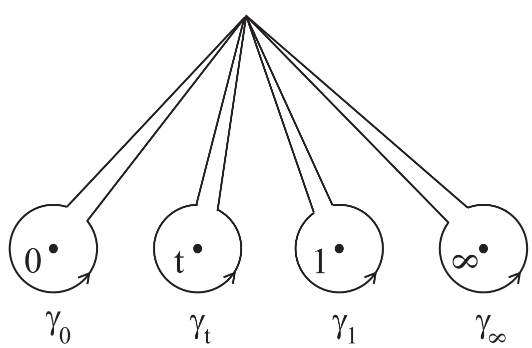

Fig. 1

As is well-known [5], isomonodromy condition for (3.1) translates into a system of matrix ODEs:

$$\frac{dA_0}{dt} = \frac{[A_t,A_0]}{t}, \qquad \frac{dA_1}{dt} = \frac{[A_t,A_1]}{t-1}. \tag{3.2}$$

where $A_{0,t,1}$ are also subject to the constraint $A_0+A_t+A_1 = -A_\infty = \text{const}$. The Lax form of these equations implies that the eigenvalues $\pm\theta_\nu$ of A_ν ($\nu = 0,t,1,\infty$) are conserved.

Now consider the function

$$\zeta(t) = (t-1)\,\mathrm{Tr}\,A_0 A_t + t\,\mathrm{Tr}\,A_t A_1. \tag{3.3}$$

Differentiating it twice with the help of (3.2), we find that

$$\zeta'(t) = \mathrm{Tr}\,A_0 A_t + \mathrm{Tr}\,A_t A_1, \tag{3.4}$$

$$\zeta''(t) = \frac{\mathrm{Tr}\,[A_0, A_t]A_1}{t(1-t)}. \tag{3.5}$$

But since for any triple of traceless 2×2 matrices $A_{0,t,1}$ one has

$$\left(\mathrm{Tr}\,[A_0, A_t]A_1\right)^2 = -2\det\begin{pmatrix} \mathrm{Tr}\,A_0^2 & \mathrm{Tr}\,A_0A_t & \mathrm{Tr}\,A_0A_1 \\ \mathrm{Tr}\,A_tA_0 & \mathrm{Tr}\,A_t^2 & \mathrm{Tr}\,A_tA_1 \\ \mathrm{Tr}\,A_1A_0 & \mathrm{Tr}\,A_1A_t & \mathrm{Tr}\,A_1^2 \end{pmatrix},$$

the relations (3.3)–(3.5) imply that the function $\zeta(t)$ satisfies the σ-form of Painlevé VI equation

$$\Big(t(t-1)\zeta''\Big)^2 = \tag{3.6}$$

$$= -2\det\begin{pmatrix} 2\theta_0^2 & t\zeta'-\zeta & \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\zeta'-\zeta & 2\theta_t^2 & (t-1)\zeta'-\zeta \\ \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\zeta'-\zeta & 2\theta_1^2 \end{pmatrix}$$

Below we will mostly work with the tau function of Painlevé VI which is related to $\zeta(t)$ by

$$\zeta(t) = t\,(t-1)\,\frac{d}{dt}\ln\tau(t). \tag{3.7}$$

The only branch points of $\tau(t)$ are 0, 1, ∞ and it can be analytically continued to the universal covering $\mathbb{P}^1\backslash\{0,1,\infty\}$.

Our aim is to construct the general solution for $\tau(t)$ in terms of monodromy of the associated linear system (3.1). The six-dimensional space of monodromy data consists of conjugacy classes of triples (M_0, M_t, M_1); note that $M_\infty = (M_1 M_t M_0)^{-1}$. It encodes four Painlevé VI parameters via

$$p_\nu = \mathrm{Tr}\,M_\nu = 2\cos 2\pi\theta_\nu, \qquad\qquad \nu = 0, t, 1, \infty, \tag{3.8}$$

and two integration constants via quadratic invariants

$$p_{\mu\nu} = 2\cos 2\pi\sigma_{\mu\nu} = \mathrm{Tr}\,M_\mu M_\nu, \qquad \mu\nu = 0t, 1t, 01. \tag{3.9}$$

The quantities (3.8)–(3.9) satisfy the relation $J(p_{0t}, p_{1t}, p_{01}) = 0$, where

$$J(p_{0t}, p_{1t}, p_{01}) = p_{0t}p_{1t}p_{01} + p_{0t}^2 + p_{1t}^2 + p_{01}^2 - \omega_{0t}p_{0t} - \omega_{1t}p_{1t} - \omega_{01}p_{01} + \omega_4 - 4, \tag{3.10}$$

with the coefficients $\omega_{0t}, \omega_{1t}, \omega_{01}, \omega_4$ given by

$$\omega_{0t} = p_0 p_t + p_1 p_\infty,$$

$$\omega_{1t} = p_t p_1 + p_0 p_\infty,$$

$$\omega_{01} = p_0 p_1 + p_t p_\infty,$$

$$\omega_4 = p_0 p_t p_1 p_\infty + p_0^2 + p_t^2 + p_1^2 + p_\infty^2.$$

The starting point of our construction is based on the Jimbo asymptotic formula [17], which relates the asymptotics of $\tau(t)$ near the critical points to monodromy data. Before we suitably reformulate this result, let us introduce a suggestive notation $\Delta_\sigma = \sigma^2$ and

$$\Delta_\nu = \frac{1}{2}\mathrm{Tr}\, A_\nu^2 = \theta_\nu^2, \qquad \nu = 0, t, 1, \infty.$$

Theorem 1. *Under assumptions that*

$$\begin{aligned} &\theta_0, \theta_t, \theta_1, \theta_\infty \notin \mathbb{Z}/2, \\ &|\mathrm{Re}\, \sigma_{0t}| < \frac{1}{2}, \quad \sigma_{0t} \neq 0, \\ &\theta_t + \epsilon\theta_0 + \epsilon'\sigma_{0t},\ \theta_1 + \epsilon\theta_\infty + \epsilon'\sigma_{0t} \notin \mathbb{Z}, \qquad \epsilon, \epsilon' = \pm 1, \end{aligned}$$

the tau function $\tau(t)$ has the following asymptotics as $t \to 0$:

$$\begin{aligned} \tau(t) = \mathrm{const} \cdot \Bigg\{ &C\left[\begin{array}{cc} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{array}; \sigma_{0t}\right] t^{\Delta_{\sigma_{0t}} - \Delta_0 - \Delta_t} \left(1 + \frac{(\Delta_{\sigma_{0t}} - \Delta_0 + \Delta_t)(\Delta_{\sigma_{0t}} - \Delta_\infty + \Delta_1)}{2\Delta_{\sigma_{0t}}} t\right) + \\ &+ C\left[\begin{array}{cc} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{array}; \sigma_{0t} - 1\right] s_{0t}^{-1} t^{\Delta_{\sigma_{0t}-1} - \Delta_0 - \Delta_t} + \\ &+ C\left[\begin{array}{cc} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{array}; \sigma_{0t} + 1\right] s_{0t}\, t^{\Delta_{\sigma_{0t}+1} - \Delta_0 - \Delta_t} + \\ &+ O\Big(\max\{t^{\Delta_{\sigma_{0t}+1} - \Delta_0 - \Delta_t + 1}, t^{\Delta_{\sigma_{0t}-1} - \Delta_0 - \Delta_t + 1}\}\Big)\Bigg\}, \end{aligned} \tag{3.11}$$

where

$$s_{0t} = \frac{(\omega_{1t} - 2p_{1t} - p_{0t}p_{01}) - (\omega_{01} - 2p_{01} - p_{0t}p_{1t})\, e^{2\pi i \sigma_{0t}}}{(2\cos 2\pi(\theta_t - \sigma_{0t}) - p_0)(2\cos 2\pi(\theta_1 - \sigma_{0t}) - p_\infty)}, \tag{3.12}$$

$$C\left[\begin{array}{cc} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{array}; \sigma\right] = \frac{\prod_{\epsilon,\epsilon'=\pm} G(1 + \theta_t + \epsilon\theta_0 + \epsilon'\sigma)\, G(1 + \theta_1 + \epsilon\theta_\infty + \epsilon'\sigma)}{\prod_{\epsilon=\pm} G(1 + 2\epsilon\sigma)},$$

and $G(z)$ denotes the Barnes G-function.

A few comments are in order:

- Two Painlevé VI integration constants correspond to the parameters σ_{0t} and s_{0t} in the asymptotic formula. The first one is present already in the leading term, whereas the second appears in subleading orders (2nd and 3rd line of (3.11)).
- Fixing p_{0t} in the relation $J(p_{0t}, p_{1t}, p_{01}) = 0$, one obtains a quadric in p_{1t}, p_{01}. The quantity s_{0t} defined by a somewhat obscure formula (3.12) is in fact a uniformizing parameter on this quadric.
- Barnes G-function may be defined as the infinite product

$$G(1+z) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z + z^2(1+\gamma)}{2}\right) \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^k \exp\left(\frac{z^2}{2k} - z\right),$$

or using analytic continuation of the integral representation

$$G(1+z) = (2\pi)^{\frac{z}{2}} \exp \int_0^\infty \frac{ds}{s} \left[\frac{1-e^{-zs}}{4\sinh^2\frac{s}{2}} - \frac{z}{s} + \frac{z^2}{2}\, e^{-s} \right], \qquad \operatorname{Re} z > -1.$$

It is essentially characterized by the recurrence relation $G(z+1) = \Gamma(z)\, G(z)$ and normalization $G(1) = 1$.

3.2. Critical expansions

As far as the leading behaviour of $\tau(t)$ is known, next terms of the critical expansion at $t = 0$ can be computed iteratively using Painlevé VI equation (3.6). The attentive reader may have already spotted a similarity between the first line of (3.11) and the first nontrivial coefficient in the conformal block expansion (2.11). This is not a mere coincidence: it was observed in [13, 14] that subsequent terms are reproduced by the ansatz

$$\tau(t) = \chi_0 \sum_{n\in\mathbb{Z}} C\left[\begin{array}{cc} \theta_1 & \theta_t \\ \theta_\infty & \theta_0 \end{array} ; \sigma_{0t}+n\right] s_{0t}^n\, \bar{\mathcal{F}}_1 \left(\Delta_0, \Delta_t, \Delta_1, \Delta_\infty, \Delta_{\sigma_{0t}+n}; t\right), \tag{3.13}$$

where $\bar{\mathcal{F}}_1$ is the $c = 1$ conformal block defined by (2.8), and χ_0 is an arbitrary constant related to the obvious inherent ambiguity of the tau function definition (3.7). Being combined with (2.7) or (2.14)–(2.15), the sum (3.13) gives an explicit series representation for the general solution of Painlevé VI.

Since the critical points $0, 1, \infty$ play completely analogous roles in Painlevé VI, one can write similar expansions at $t = 1$ and $t = \infty$. In particular,

$$\tau(t) = \chi_1 \sum_{n\in\mathbb{Z}} C\left[\begin{array}{cc} \theta_0 & \theta_t \\ \theta_\infty & \theta_1 \end{array} ; \sigma_{1t}+n\right] s_{1t}^n\, \bar{\mathcal{F}}_1 \left(\Delta_1, \Delta_t, \Delta_0, \Delta_\infty, \Delta_{\sigma_{1t}+n}; 1-t\right), \tag{3.14}$$

where s_{1t} is given by a formula analogous to (3.12):

$$s_{1t} = \frac{(\omega_{0t} - 2p_{0t} - p_{1t}p_{01}) - (\omega_{01} - 2p_{01} - p_{0t}p_{1t})\, e^{-2\pi i \sigma_{1t}}}{(2\cos 2\pi(\theta_t - \sigma_{1t}) - p_1)\,(2\cos 2\pi(\theta_0 - \sigma_{1t}) - p_\infty)}. \tag{3.15}$$

Moreover, the T-symmetry of conformal blocks

$$\mathcal{F}_c\left(\Delta_0, \Delta_t, \Delta_1, \Delta_\infty, \Delta; t\right) = (1-t)^{\Delta_0 - \Delta_t - \Delta} \mathcal{F}_c\left(\Delta_0, \Delta_t, \Delta_\infty, \Delta_1, \Delta; \frac{t}{t-1}\right)$$

yields a few more expansions of $\tau(t)$ with overlapping regions of validity (as the radius of convergence of conformal block series is believed to be 1).

As explained in [13], the expansion (3.13) can be derived from physical considerations by identifying the isomonodromic tau function with a chiral correlation function of monodromy fields associated with the regular singular points of the linear system (3.1). Moreover, such an identification is valid for arbitrary rank and any number of singularities. Monodromy fields are Virasoro primaries with conformal dimensions $\Delta_\nu = \frac{1}{2}\operatorname{Tr} A_\nu^2$. The central charge $c = 1$ in rank 2 arises from the identification of underlying CFT with $\hat{su}(2)_1$ WZW theory. The series (3.13) is nothing but the conformal expansion (2.6): the integer n labels conformal families which can occur in the OPE of two monodromy fields, and their dimensions $(\sigma_{0t} + n)^2$ are related to the conservation of monodromy.

By now the expansion (3.13) is very thoroughly tested both analytically and numerically. Solution of the recurrence relations for the coefficients of asymptotic expansion derived from Painlevé VI equation (3.6) was compared with the corresponding conformal block contributions for levels up to 10. This corresponds to checking more then 30 first terms in the expansion while keeping all parameters arbitrary. Numerical experiments with random parameter values show that critical expansions at different branch points glue together to smooth global solutions of Painlevé VI.

Conformal expansions can also be checked to arbitrary order for special values of $\vec{\theta} = (\theta_0, \theta_t, \theta_1, \theta_\infty)$ and $\vec{\sigma} = (\sigma_{0t}, \sigma_{1t}, \sigma_{01})$ for which Painlevé VI solutions may be written in a closed form. For instance, the choice $\vec{\theta} = \left(\frac14, \frac14, \frac14, \frac14\right)$ corresponds to Picard elliptic solutions [18] on Painlevé VI side and Ashkin-Teller conformal blocks [25] on the CFT side. The rest of this subsection is devoted to another example, representing a particular case of the so-called Riccati solutions.

Consider an $N \times N$ Toeplitz determinant

$$D_N(z, z'; t) = \det \left[A_{j-k}(z, z'; t)\right]_{j,k=0}^{N-1} \tag{3.16}$$

with the symbol

$$(1 + \sqrt{t}\zeta)^z (1 + \sqrt{t}\,\zeta^{-1})^{z'} = \sum_{k\in\mathbb{Z}} A_k(z, z'; t)\, \zeta^k.$$

According to Gessel's theorem [15, 23], this determinant can be interpreted as the distribution function of the first row of the random Young diagram distributed according to a z-measure. Explicitly,

$$D_N(z, z'; t) = \sum_{\lambda\in\mathbb{Y}|\lambda_1\leq N} t^{|\lambda|} \prod_{(i,j)\in\lambda} \frac{(i-j+z)(i-j+z')}{h_\lambda^2(i,j)}, \tag{3.17}$$

where $h_\lambda(i, j) = \lambda_i + \lambda'_j - i - j + 1$ denotes the hook length of the box $(i, j) \in \lambda$.

On the other hand, it is known (see e.g. [11]) that the determinant (3.16) is simply related to a Painlevé VI tau function with parameters $\vec{\theta} = \left(0, \frac{N}{2}, -\frac{N+z+z'}{2}, \frac{z-z'}{2}\right)$ and $\vec{\sigma} = \left(\frac{N}{2}, \frac{z+z'}{2}, \frac{N+z-z'}{2}\right)$:

$$\tau(t) = (1-t)^{-\frac{N(N+z+z')}{2}}\, D_N(z, z'; t).$$

It turns out [14] that in this case the expansion (3.13) contains only one conformal block and its AGT representation (2.14)–(2.15) exactly reproduces combinatorial formula (3.17).

4. Irregular limit and bilinear relations

It is clear that the proof of conformal expansion (3.13) of Painlevé VI tau function reduces to showing certain differential identities satisfied by $c = 1$ conformal blocks. Below we work out the details of this statement in a simpler situation of irregular conformal blocks of Subsection 2.4.

In the irregular limit $\Delta_{0,\infty} \to \infty$, $t \to t/(\Delta_0\Delta_\infty)$, cf (2.21), Painlevé VI equation (3.6) simplifies to

$$\left(t\zeta''\right)^2 = 4\left(\zeta'\right)^2\left(\zeta - t\zeta'\right) - 4\zeta'. \tag{4.1}$$

This is the sigma form of Painlevé III equation of type D_8 [22], which was called Painlevé III_3 in [14]. The above limit is a shortcut in a more sophisticated coalescence chain of Painlevé equations

$$P_{\mathrm{VI}} \longrightarrow P_{\mathrm{V}} \longrightarrow P_{\mathrm{III}_1} \longrightarrow P_{\mathrm{III}_2} \longrightarrow P_{\mathrm{III}_3}$$

that can be solved using conformal expansions [14].

The equation (4.1) usually appears in the physics literature in a slightly disguised form of the radial sinh-Gordon equation. Namely, if we introduce the function $\psi(r)$ by

$$e^{-2\psi(r)} = -\frac{r^2}{64}\zeta'\left(\frac{r^4}{4096}\right),$$

then it satisfies

$$\psi'' + \frac{1}{r}\psi' = \frac{1}{2}\sinh 2\psi. \tag{4.2}$$

The tau function of Painlevé III_3 is related to $\zeta(t)$ by

$$\zeta(t) = t\frac{d}{dt}\ln\tau(t).$$

Substitution of this expression in (4.1) results into a complicated 3rd order quadrilinear ODE for $\tau(t)$. However, differentiating it with respect to t, one obtains a simpler bilinear equation

$$t^3\tau\tau'''' + 4t^2\left(\tau - t\tau'\right)\tau''' + 3t^3\left(\tau''\right)^2 + 2t\left(\tau - 2t\tau'\right)\tau'' + 2\tau^2 = 0. \tag{4.3}$$

Using the dilatation generator $\delta = t\frac{d}{dt}$ and the associated Hirota derivative defined by

$$f\left(e^{\alpha}t\right)g\left(e^{-\alpha}t\right) = \sum_{k=0}^{\infty}\left(D^k f\cdot g\right)\frac{\alpha^k}{k!},$$

the equation (4.3) can be put into a more symmetric form

$$\left(D^4 + (1-2\delta)D^2 + 4t\right)\tau\cdot\tau = 0. \tag{4.4}$$

The analog of the conformal expansion (3.13) for Painlevé III_3 is written in terms of irregular conformal blocks (2.23), cf Subsection 4.2 in [14]. Namely, if we define

$$C_\sigma = \left[G(1+2\sigma)G(1-2\sigma)\right]^{-1}, \tag{4.5}$$

and, similarly to (2.8),

$$\mathcal{B}_\sigma(t) = \bar{\mathcal{F}}_1\left(\sigma^2;t\right) = t^{\sigma^2}\mathcal{F}_1\left(\sigma^2;t\right), \tag{4.6}$$

then the conjectural general solution of Painlevé III_3 is given by

$$\tau(t) = \sum_{n\in\mathbb{Z}} C_{\sigma+n}\, s^n\, \mathcal{B}_{\sigma+n}(t), \tag{4.7}$$

where the parameters σ and s are arbitrary and play the role of two integration constants. Our next task is to determine what needs to be proven to show (4.7) rigorously.

Substituting (4.7) into (4.4) and equating the coefficients of different powers of s to zero, one obtains the following infinite chain of equations

$$\sum_{m,n\in\mathbb{Z}, m+n=\ell} C_{\sigma+m}C_{\sigma+n}\left(D^4+(1-2\delta)D^2+4t\right)\mathcal{B}_{\sigma+m}\cdot\mathcal{B}_{\sigma+n} = 0, \tag{4.8}$$

which should be satisfied for any $\ell \in \mathbb{Z}$ and arbitrary complex $\sigma \notin \mathbb{Z}/2$. However, considering the shift $\sigma \to \sigma - \frac{\ell}{2}$, it becomes clear that the chain (4.8) contains only two independent identities, namely,

$$\sum_{n\in\mathbb{Z}} C_{\sigma+n}C_{\sigma-n}\left(D^4+(1-2\delta)D^2+4t\right)\mathcal{B}_{\sigma+n}\cdot\mathcal{B}_{\sigma-n} = 0 \tag{4.9}$$

and the same identity with summation index $n \in \mathbb{Z}+\frac{1}{2}$. In spite of the appearance of the Barnes functions in the structure constants (4.5), the coefficients of the bilinear relations (4.9) can be reduced to *rational* functions of σ thanks to the identity

$$\frac{C_{\sigma+n}C_{\sigma-n}}{C_\sigma^2} = \prod_{k=1-2n}^{2n-1}(2\sigma-k)^{-2(2n-|k|)}.$$

The identities (4.9) bear a striking resemblance, but are not equivalent, to blowup equations of [21]. Their more cumbersome Painlevé VI analogs can be derived in a very similar way. It should be emphasized that, although these identities would be sufficient for the proof of critical expansions (at least at the level of formal series), they can be studied without any reference to Painlevé theory. Moreover, once proved, they would provide an interesting alternative way to compute the coefficients of $c=1$ conformal block series.

5. Connection formula for Painlevé VI tau function

5.1. Recursion relations

Let us come back to Painlevé VI tau function and illustrate the usefulness of conformal expansions with a result of a different nature. As already noted above, the prefactors χ_0

and χ_1 in (3.13), (3.14) are indefinite. However, the connection coefficient

$$\chi_{01}(\vec{\theta},\vec{\sigma}) = \chi_0^{-1}\chi_1, \tag{5.1}$$

which gives relative normalization of the asymptotics of $\tau(t)$ as $t\to 0$ and $t\to 1$, is completely determined by Painlevé VI equation (3.6) and initial conditions for $\zeta(t)$.

In the applications of Painlevé equations to random matrix theory the corresponding tau functions appear as Fredholm determinants of integrable kernels defined on some interval $(a,b)\in\mathbb{R}$. In most cases their asymptotics at one of the endpoints can be easily found. It fixes the tau function normalization. The remaining (very nontrivial) problem is to compute the constant prefactor in the asymptotics at the other endpoint. Our connection coefficient (5.1) is an exact analog of this quantity in the case of generic Painlevé VI tau function, where one has no distinguished normalization.

Given $\vec{\theta}$, σ_{0t} and σ_{1t}, the value of p_{01} which enters the tau function expansions (3.13), (3.14) via s_{0t} and s_{1t}, is fixed up to the choice of solution of the equation $J\left(p_{0t},p_{1t},p_{01}\right)=0$, cf (3.10). We therefore concentrate our attention on the dependence of χ_{01} on σ_{0t} and σ_{1t}. The form of the expansions (3.13), (3.14) suggests two functional relations:

$$\frac{\chi_{01}(\sigma_{0t}+1,\sigma_{1t})}{\chi_{01}(\sigma_{0t},\sigma_{1t})} = s_{0t}^{-1}, \tag{5.2}$$

$$\frac{\chi_{01}(\sigma_{0t},\sigma_{1t}+1)}{\chi_{01}(\sigma_{0t},\sigma_{1t})} = s_{1t}, \tag{5.3}$$

where s_{0t}, s_{1t} are defined by (3.12)–(3.15). The main difficulty of the solution of (5.2)–(5.3) is hidden in the dependence of these quantities on p_{01}, as the latter depends on σ_{0t},σ_{1t} in a rather complicated way. Although the equations (5.2)–(5.3) fix $\chi_{01}(\sigma_{0t},\sigma_{1t})$ only up to periodic function of σ_{0t},σ_{1t}, this ambiguity can be resolved using explicit Painlevé VI solutions depending on continuous parameters.

We now formulate the final result of this computation, postponing the details to a future publication [16]. It is convenient to consider instead of χ_{01} the quantity

$$\bar{\chi}_{01}(\vec{\theta},\vec{\sigma}) = \chi_{01}(\vec{\theta},\vec{\sigma})\, C\Big[\begin{array}{cc}\theta_0 & \theta_t\\ \theta_\infty & \theta_1\end{array};\sigma_{1t}\Big]\Big/ C\Big[\begin{array}{cc}\theta_1 & \theta_t\\ \theta_\infty & \theta_0\end{array};\sigma_{0t}\Big]. \tag{5.4}$$

In particular, for $-\frac12<\operatorname{Re}\sigma_{0t}$, $\operatorname{Re}\sigma_{1t}<\frac12$ one can write

$$\bar{\chi}_{01}(\vec{\theta},\vec{\sigma}) = \frac{\lim_{t\to 1}(1-t)^{\theta_1^2+\theta_t^2-\sigma_{1t}^2}\tau(t)}{\lim_{t\to 0} t^{\theta_0^2+\theta_t^2-\sigma_{0t}^2}\tau(t)}. \tag{5.5}$$

Define the parameters

$$\begin{aligned}
\nu_1 &= \sigma_{0t}+\theta_0+\theta_t, & \lambda_1 &= \theta_0+\theta_t+\theta_1+\theta_\infty,\\
\nu_2 &= \sigma_{0t}+\theta_1+\theta_\infty, & \lambda_2 &= \sigma_{0t}+\sigma_{1t}+\theta_0+\theta_1,\\
\nu_3 &= \sigma_{1t}+\theta_0+\theta_\infty, & \lambda_3 &= \sigma_{0t}+\sigma_{1t}+\theta_t+\theta_\infty,
\end{aligned} \tag{5.6}$$

$$\nu_4 = \sigma_{1t} + \theta_t + \theta_1, \qquad \lambda_4 = 0,$$
$$2\nu_\Sigma = \nu_1 + \nu_2 + \nu_3 + \nu_4,$$

and let ω denote the root $z = e^{2\pi i\omega}$ of the *quadratic* equation

$$\prod_{k=1}^{4}\left(1 - ze^{2\pi i\nu_k}\right) = \prod_{k=1}^{4}\left(1 - ze^{2\pi i\lambda_k}\right), \tag{5.7}$$

explicitly given by

$$z = \frac{p_{01} + 2\cos 2\pi\left(\sigma_{0t} - \sigma_{1t}\right) - 2\cos 2\pi\left(\theta_t + \theta_\infty\right) - 2\cos 2\pi\left(\theta_0 + \theta_1\right)}{\sum_{k=1}^{4}\left(e^{2\pi i(\nu_\Sigma - \nu_k)} - e^{2\pi i(\nu_\Sigma - \lambda_k)}\right)}.$$

Then the connection coefficient (5.4)–(5.5) is given by the following conjectural evaluation:

$$\bar{\chi}_{01}(\vec{\theta}, \vec{\sigma}) = \prod_{\epsilon,\epsilon'=\pm} \frac{G(1 + \epsilon\sigma_{1t} + \epsilon'\theta_t - \epsilon\epsilon'\theta_1)G(1 + \epsilon\sigma_{1t} + \epsilon'\theta_0 - \epsilon\epsilon'\theta_\infty)}{G(1 + \epsilon\sigma_{0t} + \epsilon'\theta_t + \epsilon\epsilon'\theta_0)G(1 + \epsilon\sigma_{0t} + \epsilon'\theta_1 + \epsilon\epsilon'\theta_\infty)} \times$$
$$\times \prod_{\epsilon=\pm} \frac{G(1 + 2\epsilon\sigma_{0t})}{G(1 + 2\epsilon\sigma_{1t})} \prod_{k=1}^{4} \frac{\hat{G}(\omega + \nu_k)}{\hat{G}(\omega + \lambda_k)}, \tag{5.8}$$

where, as before, $G(\sigma)$ denotes Barnes G-function and $\hat{G}(\sigma) = \dfrac{G(1+\sigma)}{G(1-\sigma)}$. Note that the right side of (5.8) is a periodic function of ω thanks to (5.7), and therefore the solution of $z = e^{2\pi i\omega}$ may be chosen arbitrarily. It should be emphasized that we are dealing with the generic Painlevé VI tau function; two integration constants are encoded in the monodromy exponents σ_{0t}, σ_{1t}.

5.2. Algebraic example

An instructive test of the general formula (5.8) is provided by the following example. Consider 16-branch algebraic solution (Solution 30 in [19]) characterized by the parameters $\vec{\theta} = \left(\frac14, \frac14, \frac14, \frac38\right)$. It admits rational parameterization

$$\tau(t(s)) = \frac{(1 - s^4)^{-\frac18}(1 + s^4)^{-\frac{5}{192}}\left(1 + (i-1)s + is^2\right)}{(-s)^{\frac{1}{32}}(s^2 - 2s - 1)^{\frac{7}{24}}(1 - 2s - s^2)^{\frac{1}{24}}(s^4 + 6s^2 + 1)^{\frac16}} \times$$
$$\times \left[-\frac{(s^2 - i)(s^2 + 2is + 1)}{(s^2 + i)(s^2 - 2is + 1)}\right]^{\frac18}, \tag{5.9}$$

$$t(s) = \frac{(s^2 - 1)^2(s^4 + 6s^2 + 1)^3}{32s^2(s^4 + 1)^3}. \tag{5.10}$$

Pick the solution branch corresponding to $s \in (-1, 1 - \sqrt{2})$. By (5.10), this interval is in one-to-one correspondence with $t \in (0,1)$. Monodromy exponents $\vec{\sigma} = \left(\frac14, \frac16, \frac13\right)$ can be read off the tau function asymptotics near the endpoints:

$$\tau(t \to 0) = 2^{\frac{5}{64}} \cdot t^{-\frac{1}{16}}\left[1 + \frac38 e^{-\frac{i\pi}{4}} t^{\frac12} + O(t)\right], \tag{5.11}$$

$$\tau(t \to 1) = 2^{\frac{197}{576}} \cdot 3^{\frac{1}{64}} \cdot e^{i\phi} \cdot (1-t)^{-\frac{7}{72}}\left[1 + O\left((1-t)^{\frac23}\right)\right], \tag{5.12}$$

where the phase ϕ is a non-rational multiple of π explicitly given by

$$\phi = \frac{1}{8}\Big(\arctan\frac{7}{4\sqrt{2}} - \pi\Big). \tag{5.13}$$

Now if the formula (5.8) for the connection coefficient is correct (i.e. the answer coincides with the one provided by (5.11)–(5.12)), we should have the identity

$$G\left[\begin{array}{c} \frac12,\frac32,\frac13,\frac76,\frac76,\frac{5}{24},\frac{25}{24},\frac{31}{24},\frac{35}{24} \\ \frac23,\frac34,\frac34,\frac34,\frac74,\frac58,\frac58,\frac78,\frac{15}{8} \end{array}\right]\prod_{k=1}^{4}\frac{\hat{G}(\omega+\nu_k)}{\hat{G}(\omega+\lambda_k)} = 2^{\frac{19}{72}}\cdot 3^{\frac{1}{64}}\cdot e^{i\phi}, \tag{5.14}$$

where $G\left[\begin{array}{c}\alpha_1,\ldots,\alpha_m\\ \beta_1,\ldots,\beta_n\end{array}\right] = \frac{\prod_{k=1}^{m}G(\alpha_k)}{\prod_{k=1}^{n}G(\beta_k)}$ and

$$\omega = \frac{5}{48} + \frac{i}{2\pi}\,\mathrm{arctanh}\frac{1}{2+\sqrt{2}+\sqrt{3}}, \tag{5.15}$$

$$(\nu_1,\nu_2,\nu_3,\nu_4) = \left(\frac34,\frac78,\frac{19}{24},\frac23\right), \tag{5.16}$$

$$(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \left(\frac98,\frac{11}{12},\frac{25}{24},0\right). \tag{5.17}$$

Indeed, the identity (5.14) was confirmed numerically by comparison of the first 500 significant digits at both sides. Similar verifications of (5.8) have been done for more than 50 branches of about 20 exceptional algebraic Painlevé VI solutions.

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