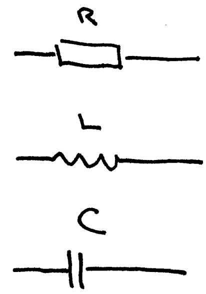
TD 6 - Correction

Exercice 1

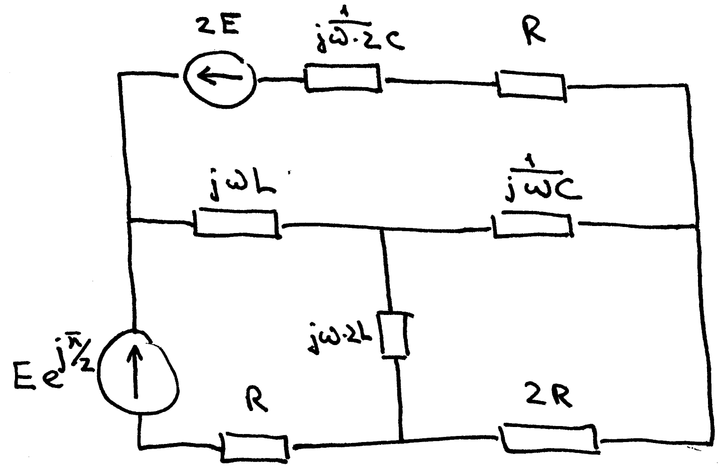


R

$j\omega L$

$\frac{1}{j\omega C}$

$\Rightarrow$



$\Downarrow$

$$e^{j\pi/2} = \underbrace{\cos \pi/2}_{0} + j \underbrace{\sin \pi/2}_{1} = j$$

$$\frac{1}{2j\omega C} = \frac{1}{2j} \cdot 2 = \frac{1}{j} = -j$$

$$\frac{1}{j\omega C} = -2j$$

$$j\omega L = j$$

$\Rightarrow$

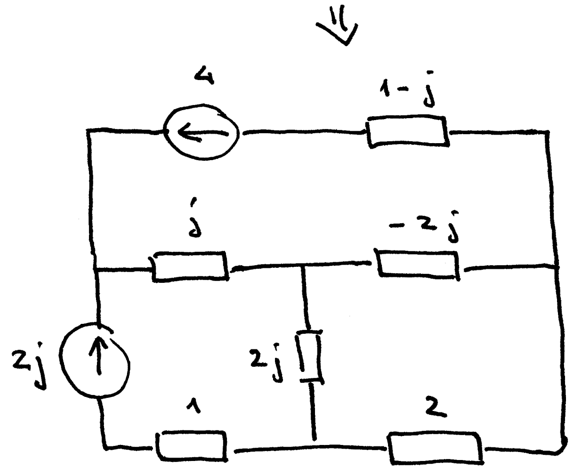
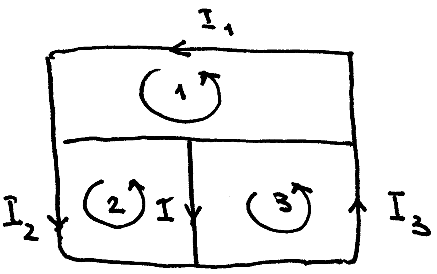


Fig. 1

La suite (qq soit la méthode) est analogue au courant contin

1.1). Courants de mailles



$$[R] = \begin{pmatrix} 1-j-2j+j & -j & 2j \\ -j & j+2j+1 & -2j \\ 2j & -2j & -2j+2+2j \end{pmatrix} =$$

$$= \begin{pmatrix} 1-2j & -j & 2j \\ -j & 1+3j & -2j \\ 2j & -2j & 2 \end{pmatrix}$$

$$[E] = \begin{pmatrix} 4 \\ -2j \\ 0 \end{pmatrix}$$

De plus $I + I_2 = I_3 \Rightarrow I = I_3 - I_2$. Donc il suffit de calculer I_2 et I_3.

Nous avons

$$\det[R] = 24 - 2j$$

$$\det\begin{pmatrix} 1-2j & 4 & 2j \\ -j & -2j & -2j \\ 2j & 0 & 2 \end{pmatrix} = 8 - 4j$$

$$\det\begin{pmatrix} 1-2j & -j & 4 \\ -j & 1+3j & -2j \\ 2j & -2j & 0 \end{pmatrix} = 20 - 20j$$

et donc

$$I = I_3 - I_2 = \frac{(20-20j)-(8-4j)}{24-2j} = \frac{12-16j}{24-2j} = \frac{16-18j}{29}$$

$$|I| = \frac{\sqrt{16^2+18^2}}{29} = \frac{2\sqrt{145}}{29}$$

$$\varphi = \operatorname{arctg}\frac{\operatorname{Im} I}{\operatorname{Re} I} = \operatorname{arctg}\left(-\frac{18}{16}\right) = \operatorname{arctg}\left(-\frac{9}{8}\right)$$

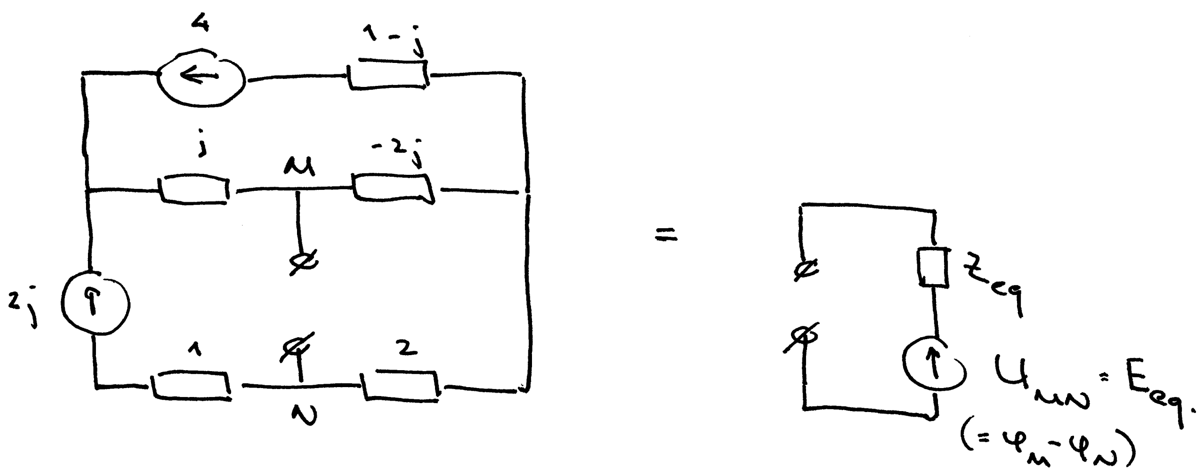
↑ déphasage p.r. à e_1

En conclusion :

$$I = \frac{2\sqrt{145}}{29}\cos\left(\omega t + \operatorname{arctg}\left(-\frac{9}{8}\right)\right).$$

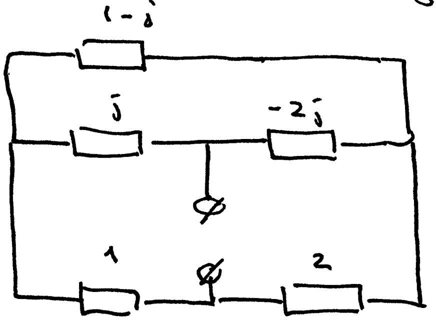
1.2). Théorème du Thévenin

D'après le théorème :



1.2.1) Calcul de Z_{eq}

En supprimant les générateurs :



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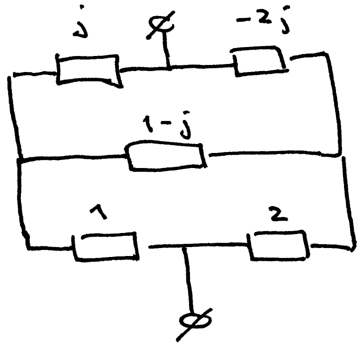

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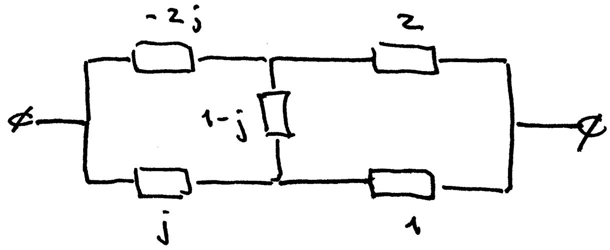
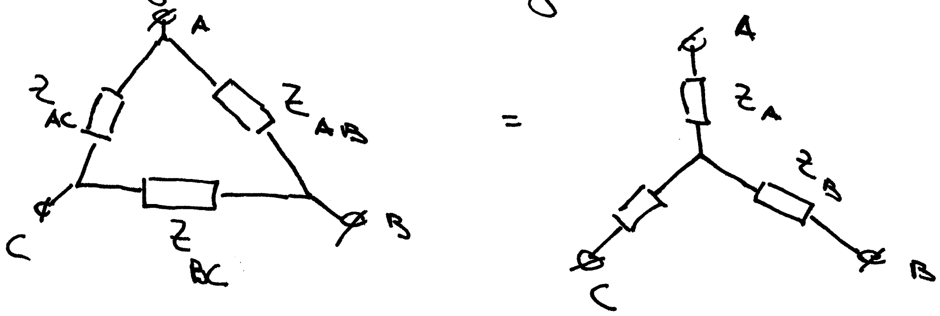


Fig. 2

Transformation triangle-étoile :



$$
\begin{cases}
(1) \quad \dfrac{Z_{AB}(Z_{AC}+Z_{BC})}{Z_{AB}+Z_{AC}+Z_{BC}} = Z_A + Z_B \\[2ex]
(2) \quad \dfrac{Z_{BC}(Z_{AC}+Z_{AB})}{Z_{AB}+Z_{AC}+Z_{BC}} = Z_B + Z_C \\[2ex]
(3) \quad \dfrac{Z_{AC}(Z_{AB}+Z_{BC})}{Z_{AB}+Z_{AC}+Z_{BC}} = Z_A + Z_C
\end{cases}
$$

$$\underset{(3)-(2)}{\Longrightarrow} \quad Z_A - Z_B = \frac{Z_{AB}(Z_{AC}-Z_{BC})}{Z_{AB}+Z_{AC}+Z_{BC}} \qquad (4)$$

$$(1)+(4) \Longrightarrow 2Z_A = \frac{2Z_{AB}Z_{AC}}{Z_{AB}+Z_{AC}+Z_{BC}} \Longrightarrow Z_A = \frac{Z_{AB}Z_{AC}}{Z_{AB}+Z_{AC}+Z_{BC}}$$

et, de façon analogue, $Z_B = \dfrac{Z_{AB}Z_{BC}}{Z_{AB}+Z_{AC}+Z_{BC}}$ et $Z_C = \dfrac{Z_{AC}Z_{BC}}{Z_{AB}+Z_{AC}+Z_{BC}}$

Nous avons besoin des formules inverses, c'est-à-dire, on veut plutôt exprimer Z_{AB}, Z_{AC}, Z_{BC} en fonction de Z_A, Z_B, Z_C

Notons que

$$\frac{Z_B}{Z_A} = \frac{Z_{BC}}{Z_{AC}} \Rightarrow Z_{AC} = \frac{Z_A}{Z_B} Z_{BC} \qquad (5)$$

$$\frac{Z_C}{Z_A} = \frac{Z_{BC}}{Z_{AB}} \Rightarrow Z_{AB} = \frac{Z_A}{Z_C} Z_{BC} \qquad (6)$$

En substituant (5) et (6) dans l'expression pour Z_A, on trouve

$$Z_A = \frac{\frac{Z_A}{Z_B} Z_{BC} \cdot \frac{Z_A}{Z_C} \cancel{Z_{BC}}}{\left(1 + \frac{Z_A}{Z_B} + \frac{Z_A}{Z_C}\right) \cancel{Z_{BC}}} = \frac{Z_A^2 Z_{BC}}{Z_A Z_B + Z_A Z_C + Z_B Z_C}$$

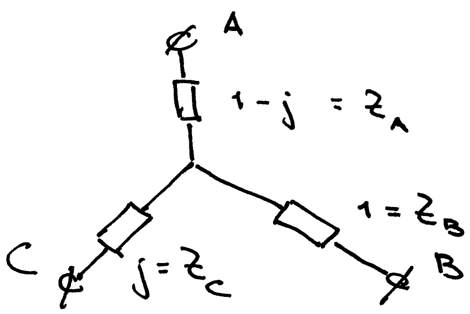
et donc

$$Z_{BC} = \frac{Z_A Z_B + Z_A Z_C + Z_B Z_C}{Z_A}$$

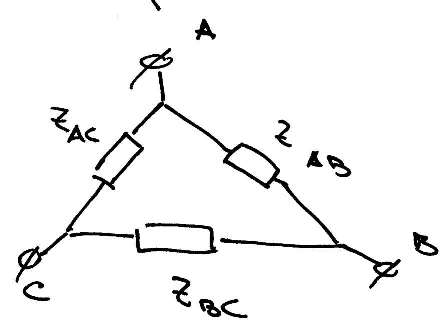
$$Z_{AC} = \frac{Z_A Z_B + Z_A Z_C + Z_B Z_C}{Z_B}$$

$$Z_{AB} = \frac{Z_A Z_B + Z_A Z_C + Z_B Z_C}{Z_C}$$

Dans notre problème, on aimerait remplacer sur la Fig. 2



par

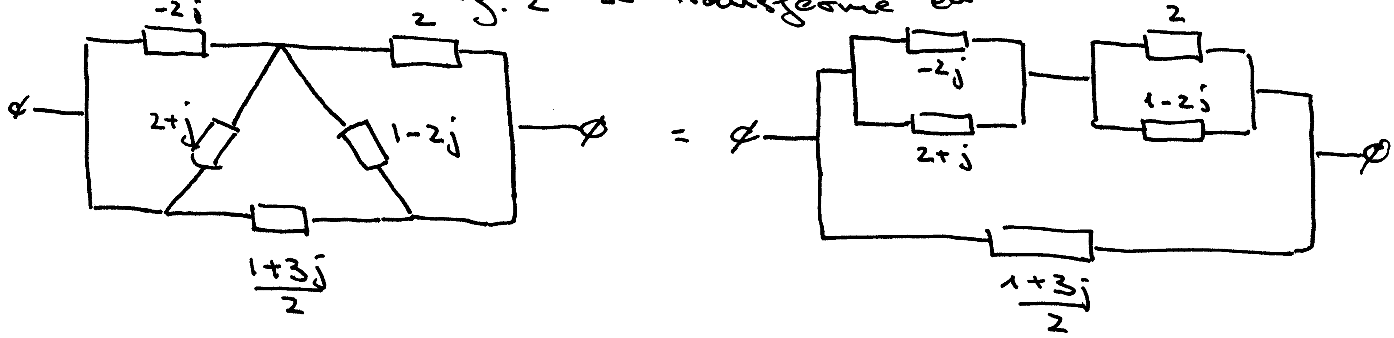


Donc:

$$Z_A Z_B + Z_A Z_C + Z_B Z_C = (1-j)\cdot 1 + (1-j)\cdot j + 1\cdot j = 2+j$$

$$Z_{BC} = \frac{2+j}{1-j} = \frac{1+3j}{2}, \quad Z_{AC} = \frac{2+j}{1} = 2+j, \quad Z_{AB} = \frac{2+j}{j} = 1-2j$$

Le schéma sur la Fig. 2 se transforme en



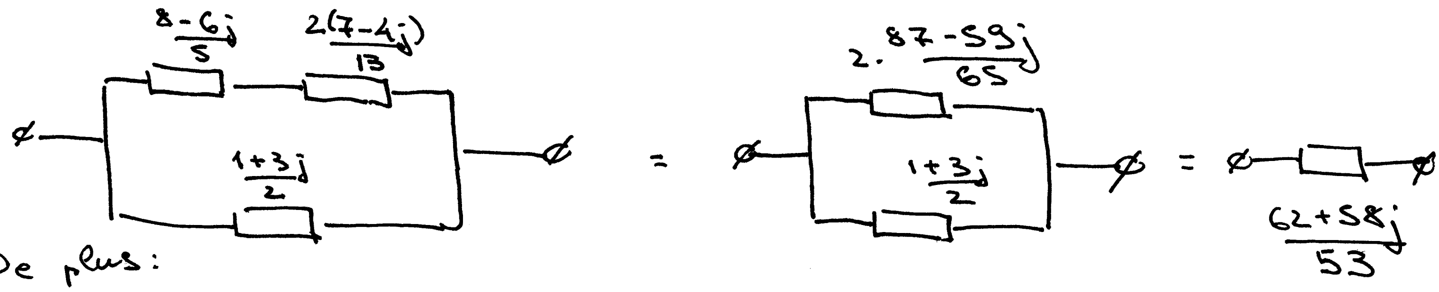
Comme

$$\frac{(-2j)(2+j)}{-2j+2+j} = \frac{2-4j}{2-j} = \frac{8-6j}{5}$$

$$\frac{2(1-2j)}{2+1-2j} = \frac{2-4j}{3-2j} = \frac{2(7-4j)}{13}$$

$$\frac{8-6j}{5} + \frac{2(7-4j)}{13} = 2\cdot\frac{87-59j}{65}$$

On obtient

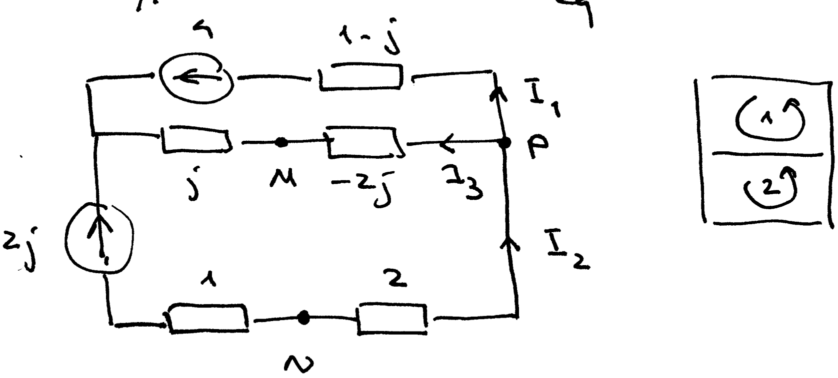


De plus :

$$\frac{2\cdot\frac{87-59j}{65}\cdot\frac{1+3j}{2}}{2\cdot\frac{87-59j}{65}+\frac{1+3j}{2}} = \frac{62+58j}{53}$$

Donc, finalement : $Z_{eq} = \frac{62+58j}{53}$

1.2.2). Calcul de E_{eq}



$$[R] = \begin{pmatrix} 1-2j & j \\ j & 3-j \end{pmatrix}$$

$$[E] = \begin{pmatrix} 4 \\ -2j \end{pmatrix}$$

$$\det[R] = 2-7j, \quad \det\begin{pmatrix} 4 & j \\ -2j & 3-j \end{pmatrix} = 10-4j, \quad \det\begin{pmatrix} 1-2j & 4 \\ j & -2j \end{pmatrix} = -4-6j$$

Donc :

$$I_1 = \frac{10-4j}{2-7j} = \frac{48+62j}{53},$$

$$I_2 = \frac{-4-6j}{2-7j} = \frac{34-40j}{53},$$

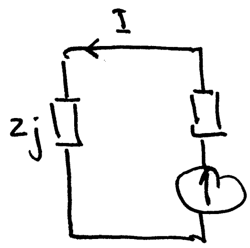
$$I_3 = I_2 - I_1 = -\frac{14+102j}{53}.$$

D'autre part :

$$\begin{cases} \varphi_P - \varphi_M = I_3(-2j) \\ \varphi_N - \varphi_P = I_2 \cdot 2 \end{cases} \Rightarrow \varphi_M - \varphi_N = -I_3(-2j) - I_2 \cdot 2 =$$

$$= -\frac{14+102j}{53} \cdot 2j - \frac{34-40j}{53} \cdot 2 = \frac{136+52j}{53}$$

1.2.3). Calcul de I.



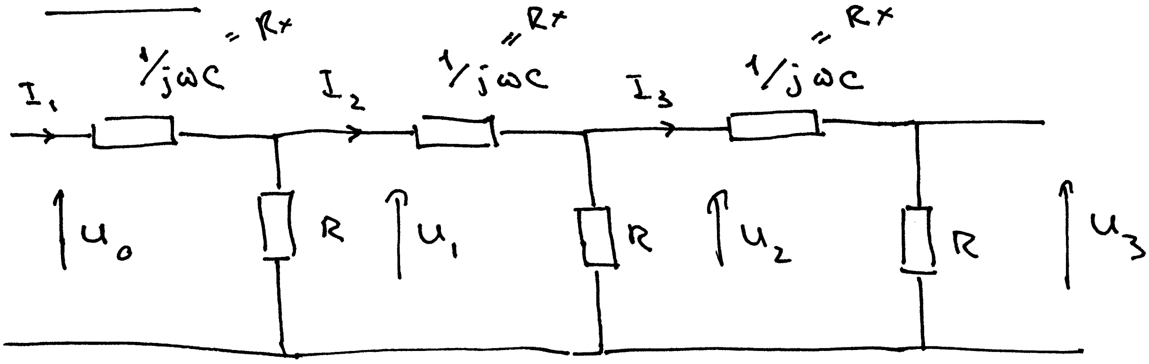
$$Z_{eq} = \frac{62+58j}{53}$$

$$E_{eq} = \frac{136+52j}{53}$$

$$I = \frac{E_{eq}}{2j + Z_{eq}} = \frac{\frac{136+52j}{53}}{2j + \frac{62+58j}{53}} = \frac{16-18j}{29}$$

On retrouve donc le résultat de la méthode de mailles.

Exercice 2



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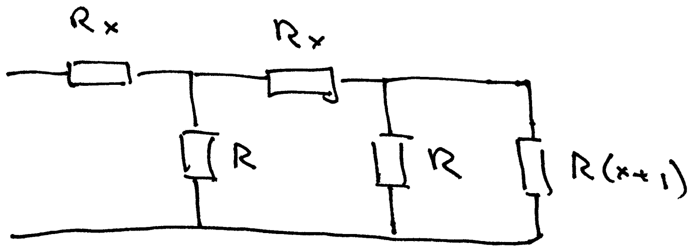

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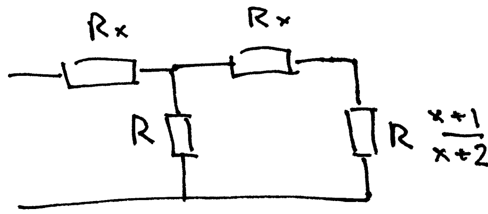

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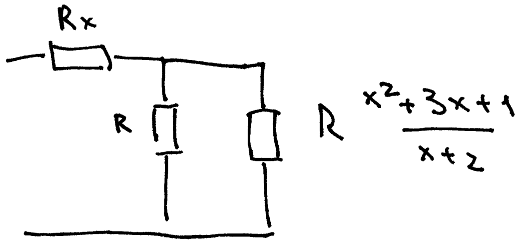

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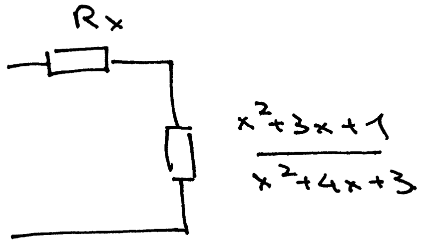

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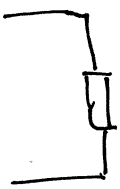


$$\frac{x^3+5x^2+6x+1}{x^2+4x+3}R = Z_e$$ (2) (impédance d'entrée)

Donc $$I_1 = \frac{U_0}{Z_e} = \frac{x^2+4x+3}{x^3+5x^2+6x+1}\frac{U_0}{R}$$

$$U_1 = U_0 - I_1 Rx = U_0 - \frac{x(x^2+4x+3)}{x^3+5x^2+6x+1}U_0 =$$

$$= \frac{x^2+3x+1}{x^3+5x^2+6x+1}U_0$$

$$I_2 = I_1 - \frac{U_1}{R} = \frac{I_1R - U_1}{R} = \left(\frac{x^2+4x+3}{x^3+5x^2+6x+1} - \frac{x^2+3x+1}{x^3+5x^2+6x+1}\right)\frac{U_0}{R}$$

$$\frac{x+2}{x^3+5x^2+6x+1}\frac{U_0}{R}$$

$$U_2 = U_1 - I_2 R x = \frac{x^2+3x+1}{x^3+5x^2+6x+1} U_0 - \frac{(x+2)\,x}{x^3+5x^2+6x+1} U_0$$

$$= \frac{x+1}{x^3+5x^2+6x+1} U_0$$

$$I_3 = I_2 - \frac{U_2}{R} = \left(\frac{x+2}{x^3+5x^2+6x+1} - \frac{x+1}{x^3+5x^2+6x+1} \right) \frac{U_0}{R} =$$

$$= \frac{1}{x^3+5x^2+6x+1} \frac{U_0}{R}$$

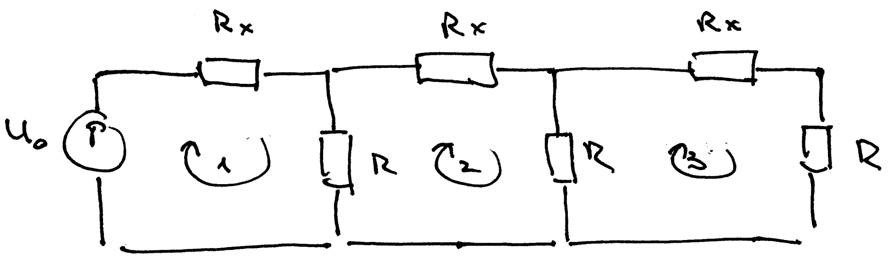
$$U_3 = U_2 - I_3 R x = \left(\frac{x+1}{x^3+5x^2+6x+1} - x \frac{1}{x^3+5x^2+6x+1} \right) U_0$$

$$= \frac{1}{x^3+5x^2+6x+1} U_0$$

L'amplification en tension est donc

(1) $$A_v = \frac{1}{x^3+5x^2+6x+1}$$

2ème méthode



$$[R] = \begin{pmatrix} R(1+x) & -R & 0 \\ -R & R(2+x) & -R \\ 0 & -R & R(2+x) \end{pmatrix}, \quad [E] = \begin{pmatrix} U_0 \\ 0 \\ 0 \end{pmatrix}$$

$$\det[R] = R^3\left((1+x)(2+x)^2 - (1+x) - (2+x) \right) = R^3\left((1+x)(x^2+4x+4) - 2x - 3 \right) =$$

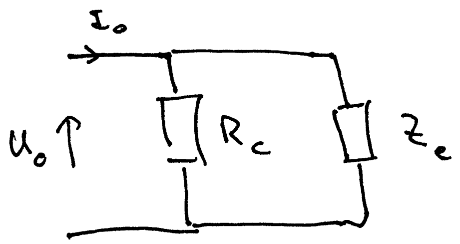
$$= R^3 \left(x^3 + 5x^2 + 6x + 1 \right)$$

$$\det \begin{pmatrix} R(1+x) & -R & U_0 \\ -R & R(2+x) & 0 \\ 0 & -R & 0 \end{pmatrix} = U_0 R^2 \Rightarrow I_3 = \frac{U_0 R^2}{R^3 (x^3+5x^2+6x+1)}$$

$$U_3 = I_3 R = \frac{U_0}{x^3+5x^2+6x+1} \Longrightarrow A_v = \frac{1}{x^3+5x^2+6x+1}$$

(on retrouve donc le même résultat)

3). On obtient



Résistance totale :

$$\frac{R_c Z_e}{R_c + Z_e}$$

et donc

$$I_0 = \frac{U_0 (R_c + Z_e)}{R_c Z_e} = U_0\left(\frac{1}{Z_e} + \frac{1}{R_c}\right) =$$

$$= U_0\left(\frac{1}{R_c} + \frac{(x^3+5x^2+6x+1)^{-1}}{R(x^2+4x+3)^{-1}}\right)$$

Cela implique

$$A_i = \frac{I_3}{I_0} = \frac{U_0}{R} \cdot \frac{1}{x^3+5x^2+6x+1} \Big/ \frac{U_0}{R}\left(\frac{R}{R_c} + \frac{(x^3+5x^2+6x+1)^{-1}}{(x^2+4x+3)^{-1}}\right) =$$

$$= \frac{1}{\frac{R}{R_c}(x^3+5x^2+6x+1) + x^2+4x+3}$$

4). Supposons que $\arg(A_v) = \pi$. En particulier, A_v doit être un nombre réel. D'autre part $x = \frac{1}{jRC\omega} = -jy$, $y = \frac{1}{RC\omega}$, avec $y \in \mathbb{R}_+$. Donc

$$A_v^{-1} = x^3+5x^2+6x+1 = (-jy)^3 + 5(-jy)^2 + 6(-jy) + 1 =$$

$$= jy^3 - 5y^2 - 6jy + 1 = 1 - 5y^2 + jy(y^2-6)$$

et on obtient 2 conditions

$$\begin{cases} y(y^2-6) = 0 \\ 1 - 5y^2 < 0 \end{cases} \Longrightarrow y = 0,\ y = \sqrt{6},\ y = -\sqrt{6}$$

($y = 0$: ne vérifie pas $1-5y^2<0$; $y = -\sqrt{6}$: ne marche pas car $y>0$)

D'où $y = \sqrt{6}$, $\omega_1 = \frac{1}{\sqrt{6}\,RC}$.

Nous avons également

$$A_v^{-1} = 1-5y^2 = 1-5\cdot 6 = -29 \Rightarrow A_v = -\frac{1}{29}$$

$$Z_e = \frac{x^3+5x^2+6x+1}{x^2+4x+3} R = \frac{A_v^{-1} R}{x^2+4x+3} = \frac{-29 R}{-6+4(-j\sqrt{6})+3} = \frac{29 R}{3-4\sqrt{6}j}$$

5). Pour A_i réelle, A_i^{-1} est réelle également. Donc

$$\underbrace{\frac{R}{R_c}\left(1-5y^2+jy(y^2-6)\right)}_{= x^3+5x^2+6x+1} + \underbrace{\left(-y^2-4jy+3\right)}_{= x^2+4x+3} \quad \text{doit} \in \mathbb{R}$$

Donc

$$\frac{R}{R_c} y(y^2-6) - 4y = 0 \Rightarrow y^2 - 6 - \frac{4R_c}{R} = 0$$

$$y = \sqrt{6+\frac{4R_c}{R}}$$

$$\omega_2 = \frac{1}{RC\sqrt{6+\frac{4R_c}{R}}}$$

En supposant par la suite $\omega = \omega_2$, on a

$$A_i^{-1} = \frac{R}{R_c}(1-5y^2) + (3-y^2) = \frac{R}{R_c}\left(1-5\left(6+\frac{4R_c}{R}\right)\right) + 3 - \left(6+\frac{4R_c}{R}\right) =$$

$$= -29\frac{R}{R_c} - 20 - 3 - 4\frac{R_c}{R} =$$

$$= -4\frac{R_c}{R} - 29\frac{R}{R_c} - 23$$

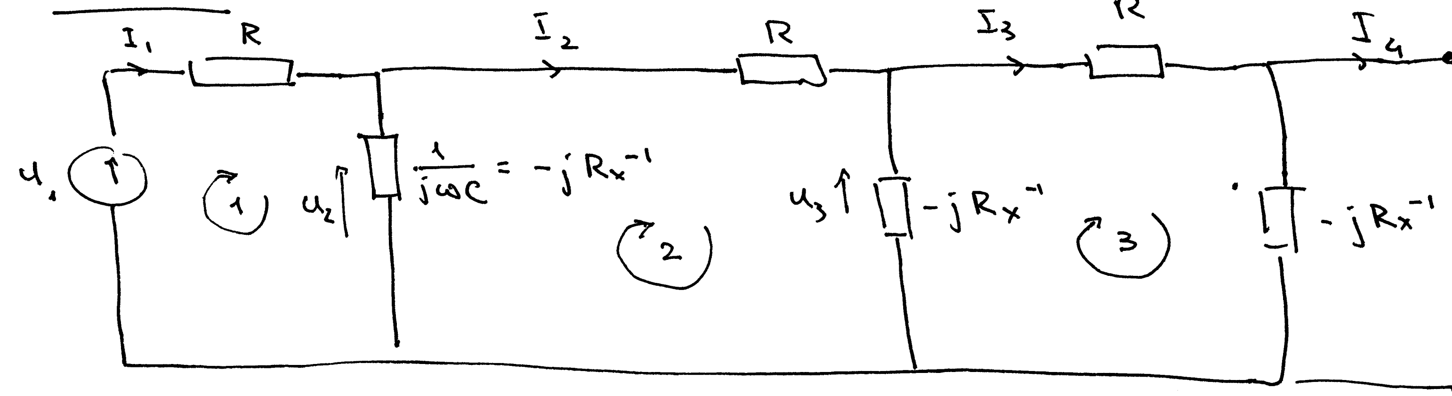
Si l'on introduit $\alpha = \frac{R_c}{R}$, alors

$$A_i^{-1} = -4\alpha - 29\alpha^{-1} - 23 = -\left(2\sqrt{\alpha} - \frac{\sqrt{29}}{\sqrt{\alpha}}\right)^2 - 4\sqrt{29} - 23$$

Il est donc clair que la valeur minimale de $|A_i^{-1}|$ (et donc la valeur maximale de $|A_i|$) correspond à $\alpha = \frac{\sqrt{29}}{2} = \frac{R_c}{R}$.

La valeur de $|A_i|_{max}$ est alors $\frac{1}{23+4\sqrt{29}}$.

Exercice 3



1). Nous avons

$$U_1 = U_2 + I_1 R \qquad\qquad U_1 = U_2 + I_2 R + jxU_2$$

$$I_2 = I_1 - \frac{U_2}{-jRx^{-1}} \Rightarrow I_1 = I_2 + jx\frac{U_2}{R}$$

Sous forme matricielle:

$$\begin{pmatrix} I_1 \\ U_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & j\frac{x}{R} \\ R & 1+jx \end{pmatrix}}_{\text{matrice de transfert } T_1} \begin{pmatrix} I_2 \\ U_2 \end{pmatrix}$$

2). Nous avons, par répétition

$$\begin{pmatrix} I_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} 1 & j^x/_R \\ R & 1+jx \end{pmatrix}\begin{pmatrix} I_2 \\ U_2 \end{pmatrix} = \begin{pmatrix} 1 & j^x/_R \\ R & 1+jx \end{pmatrix}\begin{pmatrix} 1 & j^x/_R \\ R & 1+jx \end{pmatrix}\begin{pmatrix} I_3 \\ U_3 \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 1 & j^x/_R \\ R & 1+jx \end{pmatrix}\begin{pmatrix} 1 & j^x/_R \\ R & 1+jx \end{pmatrix}\begin{pmatrix} 1 & j^x/_R \\ R & 1+jx \end{pmatrix}}_{T}\begin{pmatrix} I_4 \\ U_4 \end{pmatrix}$$

Donc

$$T = \begin{pmatrix} 1+jx & 2j^x/_R - \frac{x^2}{R} \\ 2R+jxR & 1+3jx-x^2 \end{pmatrix}\begin{pmatrix} 1 & j^x/_R \\ R & 1+jx \end{pmatrix} =$$

$$= \begin{pmatrix} 1-x^2+3jx & (1+jx)(3j-x)\frac{x}{R} \\ (3+4jx-x^2)R & 1-5x^2+6jx-jx^3 \end{pmatrix}$$

Dans le cas non-chargé : $I_4 = 0$, donc

$$\begin{pmatrix} I_1 \\ U_1 \end{pmatrix} = T \begin{pmatrix} 0 \\ U_4 \end{pmatrix} = \begin{pmatrix} (1+jx)(3j-x)\frac{x}{R} U_4 \\ [1-5x^2+6jx-jx^3] U_4 \end{pmatrix}$$

et, par conséquent :

$$H = \frac{U_4}{U_1} = \frac{1}{1-5x^2+6jx-jx^3} .$$

3). ⓐ $\varphi(x) = \arg\left(\frac{1}{1-5x^2+6jx-jx^3}\right) =$

$= -\arg\left(1-5x^2+j(6x-x^3)\right) =$

$= -\operatorname{arctg} \frac{6x-x^3}{1-5x^2}$

ⓑ 1). opposition de phase $\Rightarrow$ $\varphi(x) = \pi$ $\Rightarrow$ $1-5x^2+6jx-jx^3$ réel négatif

d'où les conditions :

$$\begin{cases} 6x - x^3 = 0 \\ 1 - 5x^2 < 0 \end{cases} \Rightarrow x = \sqrt{6} \quad \left(\Rightarrow \omega = \frac{\sqrt{6}}{RC}\right)$$

le gain :

$$G = \left|\frac{U_4}{U_1}\right| = \frac{1}{|1-5x^2|} = \frac{1}{29} .$$

2). quadrature $\Rightarrow$ $\varphi(x) = \pm i\pi/2$ $\Rightarrow$ $1-5x^2+6jx-jx^3 =$

$= \left|\frac{U_1}{U_4}\right| e^{\mp i\pi/2} = \left|\frac{U_1}{U_4}\right|(\mp i)$

Alors, dans ce cas

$$1-5x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{5}} \quad \left(\Rightarrow \omega = \frac{1}{\sqrt{5}RC}\right)$$

et pour le gain

$$G = \frac{1}{|6x-x^3|} = \frac{1}{\frac{1}{\sqrt{5}}\left(6-\frac{1}{5}\right)} = \frac{5\sqrt{5}}{29} .$$